

Solution Set #7

Transform the given equation or system into an equivalent system of first-order differential equations.

4.1.1. $x'' + 3x' + 7x = t^2$

Equivalent system:

~~$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_2' &= -3x_1 - 7x_2 + t^2 \end{aligned}$$~~

$$\begin{aligned} x_1 &= x \\ x_1' &= -3x_1 - 7x_2 + t^2 \end{aligned}$$

4.1.3.

$$t^2 x'' + t x' + (t^2 - 1)x = 0$$

~~$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ t^2 x_2' &= -t x_1 - (t^2 - 1)x_2 \end{aligned}$$~~

$$\begin{aligned} x_1 &= x \\ t^2 x_1' &= -t x_1 - (t^2 - 1)x_2 \end{aligned}$$

4.1.13

Find a general solution to the ODE or system of ODEs, and use a graphing calculator to construct a direction field and typical solution curves

$$\begin{aligned} x' &= -2y & x(0) &= 1 \\ y' &= 2x & y(0) &= 0 \end{aligned}$$

$$x'' = -2y' = -4x$$

$$\Rightarrow x'' + 4x = 0$$

$$\Rightarrow x(t) = A \cos(2t) + B \sin(2t)$$

and

$$y = -\frac{1}{2}x'(t) = -\frac{1}{2}(-2A \sin(2t) + 2B \cos(2t)) \\ = A \sin(2t) - B \cos(2t).$$

$$\Rightarrow y(t) = A \sin(2t) - B \cos(2t)$$

Now, solving for A and B:

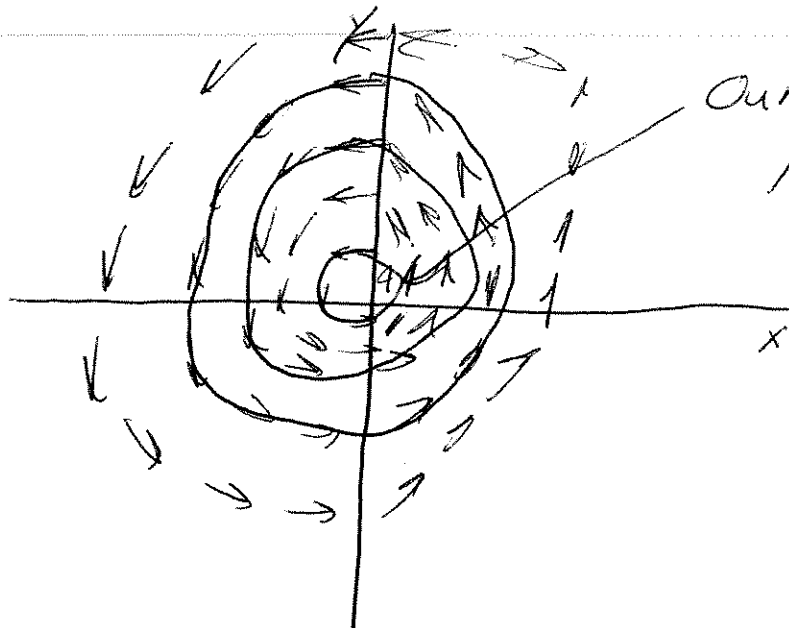
$$x(0) = A = 1$$

$$y(0) = -B = 0$$

$$\Rightarrow x(t) = \cos(2t)$$

$$y(t) = \sin(2t)$$

Direction Field



Our Solution

Note: should be circles. I'm not the best artist.

7, 1.22.

a) Beginning with the general solution of the system $x' = -2y$, $y' = 2x$ of problem 13, calculate $x^2 + y^2$ to show that the trajectories are circles.

$$\begin{aligned}x(t) &= A \cos(2t) + B \sin(2t) \\ y(t) &= A \sin(2t) - B \cos(2t)\end{aligned}$$

$$\begin{aligned}x^2(t) + y^2(t) &= A^2 \cos^2(2t) + 2AB \sin(2t) \cos(2t) + B^2 \sin^2(2t) \\ &\quad + A^2 \sin^2(2t) - 2AB \sin(2t) \cos(2t) + B^2 \cos^2(2t) \\ &= A^2 + B^2\end{aligned}$$

So, the trajectory is a circle of radius $\sqrt{A^2 + B^2}$, where $A = x(0)$ and $B = -y(0)$.

b) Show similarly that the trajectories of the system $x' = \frac{1}{2}y$, $y' = -8x$ of problem 15 are ellipses with equations of the form $16x^2 + y^2 = C^2$.

$$x' = \frac{1}{2}y, \quad y' = -8x$$

has solution:

$$x'' = \frac{1}{2}y' = -4x \Rightarrow x'' + 4x = 0$$

$$\text{So, } x(t) = A \cos(2t) + B \sin(2t) \text{ and}$$

$$y = 2x'(t) = -4A \sin(2t) + 4B \cos(2t)$$

So,

$$x(t)^2 = A^2 \cos^2(2t) + 2AB \cos(2t) \sin(2t) + B^2 \sin^2(2t)$$

$$y(t)^2 = 16A^2 \sin^2(2t) - 16AB \cos(2t) \sin(2t) + 16B^2 \cos^2(2t)$$

which gives us the relation:

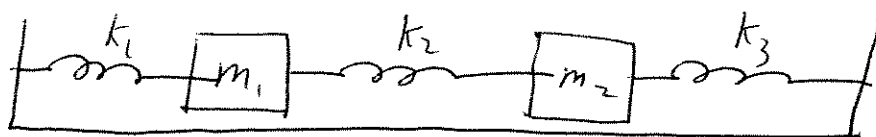
$$16x^2(t) + y^2(t) = 16(A^2 + B^2) = C^2$$

An ellipse.

4.1.24 Derive the equations

$$\begin{aligned} m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 \end{aligned}$$

for the displacements (from equilibrium) of the two masses:



If we view x_1 as the displacement of m_1 from equilibrium and x_2 as the displacement of m_2 from equilibrium then Hooke's law tells us:

$$\begin{aligned} F_1 &= -k_1 x_1 + k_2(x_2 - x_1) \\ &\text{and} \\ F_2 &= -k_2(x_2 - x_1) - k_3 x_2 \end{aligned}$$

and Newton's second law, $F=ma$, tells us:

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' &= -k_2(x_2 - x_1) - k_3 x_2 \end{aligned}$$

or, in a more compact form,

$$\begin{aligned} m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 \end{aligned}$$

t.2.1

Find the general solution and construct a direction field and draw some typical solutions.

$$\begin{aligned}x' &= -x + 3y \\ y' &= 2y\end{aligned}$$

Well, $y' = 2y$ has solution $y(t) = Ae^{2t}$

So,

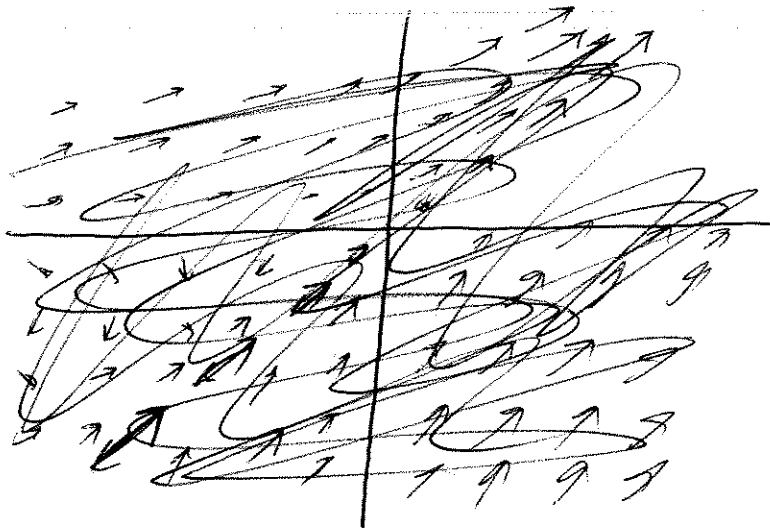
$$\begin{aligned}x' &= -x + 3Ae^{2t} \\ \Rightarrow x' + x &= 3Ae^{2t}\end{aligned}$$

which has integration factor $p(x) = e^{\int dx} = e^x$

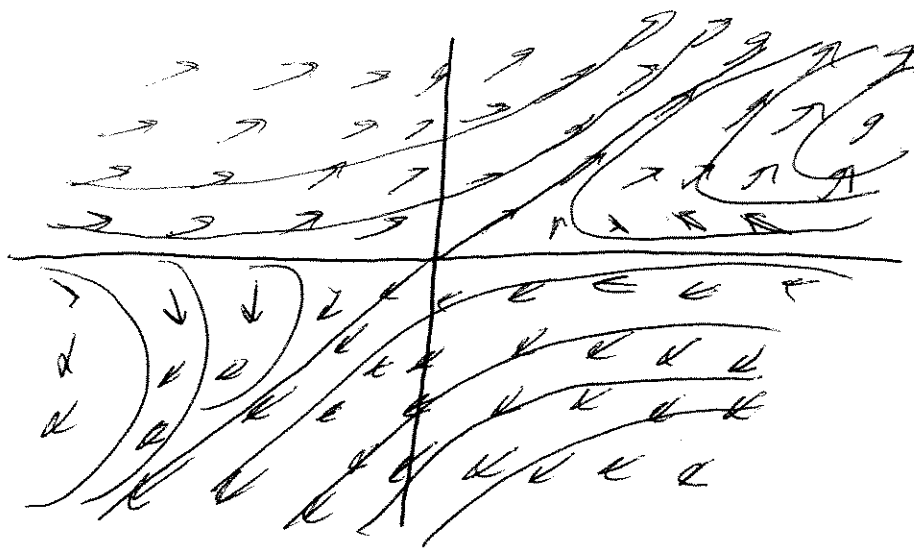
So,

$$\begin{aligned}\frac{d}{dt}(e^t x) &= 3Ae^{3t} \\ \Rightarrow e^t x &= \frac{3A}{3} e^{3t} + B \Rightarrow x(t) = Ae^{2t} + Be^{-t}\end{aligned}$$

The direction field looks like:



Sorry.



4.2.10

$$\begin{aligned} x' + 2y' &= 4x + 5y & x(0) &= 1 \\ 2x' - y' &= 3x & y(0) &= -1 \end{aligned}$$

$$\Rightarrow \begin{aligned} 5x' &= 10x + 5y \\ -5y' &= -5x - 10y \Rightarrow 5y' = 5x + 10y \end{aligned}$$

$$\text{So, } 5x'' = 10x' + 5y' = 10x' + (5x + 10y)$$

$$\Rightarrow 5x'' = 10x' + (5x + 10x' - 20x)$$

$$\Rightarrow 5x'' = 20x' - 15x$$

$$\Rightarrow x'' = 4x' - 3x \quad \text{So,}$$

$$x'' - 4x' + 3x = 0$$

has characteristic equation

$$r^2 - 4r + 3 = (r-3)(r-1)$$

which has roots $r = 3, 1$

$$x(t) = c_1 e^{3t} + c_2 e^t$$

and

$$y' = 2x' - 3x = 3c_1 e^{3t} - c_2 e^t$$

$$\Rightarrow y(t) = c_1 e^{3t} - c_2 e^t$$

Which has particular solution:

$$\begin{aligned}x(0) &= c_1 + c_2 = 1 \\ y(0) &= c_1 - c_2 = -1\end{aligned}$$

$$\Rightarrow \begin{aligned}2c_1 &= 0 \Rightarrow c_1 = 0 \\ c_2 &= 1\end{aligned}$$

So,

$$\boxed{\begin{aligned}x(t) &= e^t \\ y(t) &= -e^t\end{aligned}}$$

1.2.19

$$\begin{aligned}x' &= 4x - 2y \\ y' &= -4x + 4y - 2z \\ z' &= -4y + 4z\end{aligned}$$

$$x'' = 4x' - 2y' = 4x' - 2(-4x + 4y - 2z)$$

$$x'' = 4x' + 8x - 8y + 4z$$

$$\Rightarrow x^{(3)} = 4x'' + 8x' - 8y' + 4z' = 4x'' + 8x' - 8y' + 4(-4y + 4z)$$

$$\Rightarrow x^{(3)} = 4x'' + 8x' - 8y' - 16y + 16z$$

$$\Rightarrow x^{(3)} = 4x'' + 8x' - 8y' - 16y + 8(-y' - 4x + 4y)$$

$$\Rightarrow x^{(3)} = 4x'' + 8x' - 16y' + 16y - 32x$$

$$\Rightarrow x^{(3)} = 4x'' + 8x' - 8(4x' - x'') + 8(4x - x') - 32x$$

$$\Rightarrow x^{(3)} = 12x'' - 32x'$$

$$\Rightarrow x^{(3)} - 12x'' + 32x' = 0$$

the characteristic equation is:

$$r(r-8)(r-4)$$

with roots $r = 0, 4, 8$.

So,

$$x(t) = c_1 + c_2 e^{4t} + c_3 e^{8t}$$

$$x'(t) = 4c_2 e^{4t} + 8c_3 e^{8t}$$

$$y(t) = 2x - \frac{1}{2}x' = 2c_1 - 2c_3 e^{8t}$$

and

$$z(t) = -2x(t) + 2y(t) - \frac{1}{2}y'(t)$$

$$= -2c_1 - 2c_2 e^{4t} - 2c_3 e^{8t} + 4c_1 - 4c_3 e^{8t} + 8c_3 e^{8t}$$

$$\Rightarrow z(t) = 2c_1 - 2c_2 e^{4t} + 2c_3 e^{8t}$$

So,

$$\begin{array}{l} x(t) = c_1 + c_2 e^{4t} + c_3 e^{8t} \\ y(t) = 2c_1 - 2c_3 e^{8t} \\ z(t) = 2c_1 - 2c_2 e^{4t} + 2c_3 e^{8t} \end{array}$$

1-2.28

Calculate the operational determinant of the given system in order to determine how many arbitrary constants should appear in a general solution. Then solve the system explicitly to find a general solution, if possible.

~~opera~~

$$\begin{array}{l} (D^2 + D)x + D^2 y = 2e^{-t} \\ (D^2 - 1)x + (D^2 - D)y = 0 \end{array}$$

Operational determinant:

$$(D^2 + D)(D^2 - D) - D^2(D^2 - 1) = D^4 - D^3 + D^3 - D^2 - D^4 + D^2 = 0$$

So, there are 0 (!) arbitrary constants.
How is this possible? Well, if we
look at our system we get the relation:

$$(D+1)x + Dy = 2e^{-t}$$

$$\Rightarrow Dy = 2e^{-t} - (D+1)x$$

$$\Rightarrow D^2y = -2e^{-t} - D^2x$$

$$\Rightarrow D^2x + D^2y = -2e^{-t}$$

Plugging this into the ~~first~~^{second} relation we get

$$Dx - 2e^{-t} = 2e^{-t}$$

$$\Rightarrow Dx = 4e^{-t} \Rightarrow x(t) = -4e^{-t} + C_1$$

the second relation gives us

$$-2e^{-t} - (-4e^{-t} + C_1) - Dy = 0$$

$$\Rightarrow Dy = 2e^{-t} - C_1$$

$$\Rightarrow y(t) = -2e^{-t} - C_1t + C_2$$

~~$$D^2x = -4e^{-t}$$~~

~~$$Dx = 4e^{-t}$$~~

~~$$D^2y = -2e^{-t}$$~~

~~$$Dy = 2e^{-t} - C_1$$~~

~~$$-4e^{-t} + 4e^{-t}$$~~

However, this cannot be,
as $D^2x + D^2y = -6e^{-t}$
which contradicts our earlier
relation. So, there is
no solution.

4.2-30

Suppose that the salt concentration in each of the two brine tanks of example 2 of section 4.1 initially ($t=0$) is 0.5 lb/gal. Then solve the system in Eq. 5 there to find the amounts $x(t)$ and $y(t)$ of salt in the two tanks at time t .

The system in Eq. 5 is:

$$\begin{aligned} 20x' &= -6x + y \\ 20y' &= 6x - 3y \end{aligned}$$

$$\begin{aligned} \Rightarrow 20x'' &= -6x' + y' \\ \Rightarrow 400x'' &= -120x' + 20y' \\ \Rightarrow 400x'' &= -120x' + 6x - 3y \\ \Rightarrow 400x'' &= -120x' + 6x - 3(20x' + 6x) \\ \Rightarrow 400x'' &= -120x' + 6x - 60x' - 18x \\ \Rightarrow 400x'' + 180x' + 12x &= 0 \end{aligned}$$

which has characteristic equation =

$$400r^2 + 180r + 12 = 0$$

$$r = \frac{-180 \pm \sqrt{180^2 - 4(400)(12)}}{2(400)}$$

$$= -\frac{9}{40} \pm \frac{\sqrt{33}}{40} = \frac{-9 \pm \sqrt{33}}{40}$$

So,

$$x(t) = c_1 e^{-\frac{9+\sqrt{33}}{40}t} + c_2 e^{-\frac{9-\sqrt{33}}{40}t}$$

$$y(t) = 20x' + 6x$$

\Rightarrow

$$\begin{aligned}y(t) &= c_1 \left(\frac{-9 + \sqrt{33}}{2} \right) e^{-\frac{9 + \sqrt{33}}{40}t} + c_2 \left(\frac{-9 - \sqrt{33}}{2} \right) e^{-\frac{9 - \sqrt{33}}{40}t} \\ &\quad + 6c_1 e^{-\frac{9 + \sqrt{33}}{40}t} + 6c_2 e^{-\frac{9 - \sqrt{33}}{40}t} \\ &= c_1 \left(\frac{9 + \sqrt{33}}{2} \right) e^{-\frac{9 + \sqrt{33}}{40}t} + c_2 \left(\frac{9 - \sqrt{33}}{2} \right) e^{-\frac{9 - \sqrt{33}}{40}t}\end{aligned}$$

So, given initial conditions:

$$\begin{aligned}x(0) &= c_1 + c_2 = -5 \\ y(0) &= c_1 \left(\frac{9 + \sqrt{33}}{2} \right) + c_2 \left(\frac{9 - \sqrt{33}}{2} \right) = .5\end{aligned}$$

Solving this system we get

$$c_2 \left(\frac{9 - \sqrt{33}}{2} \right) - c_2 \left(\frac{9 + \sqrt{33}}{2} \right) = .5 - .5 \left(\frac{9 + \sqrt{33}}{2} \right)$$

$$\Rightarrow -\sqrt{33} c_2 = -\left(\frac{7 + \sqrt{33}}{4} \right)$$

$$\Rightarrow c_2 = \frac{7 + \sqrt{33}}{4\sqrt{33}} = \frac{33 + 7\sqrt{33}}{132}$$

and

$$c_1 = \frac{1}{2} - c_2 = \frac{+33 - 7\sqrt{33}}{132}$$

So,

$$x(t) = \left(\frac{33 - 7\sqrt{33}}{132} \right) e^{-\frac{9+\sqrt{33}}{40}t} + \left(\frac{33+7\sqrt{33}}{132} \right) e^{-\frac{9-\sqrt{33}}{40}t}$$

and

$$y(t) = \left(\frac{33-7\sqrt{33}}{132} \right) \left(\frac{9+\sqrt{33}}{2} \right) e^{-\frac{9+\sqrt{33}}{40}t} + \left(\frac{33+7\sqrt{33}}{132} \right) \left(\frac{9-\sqrt{33}}{2} \right) e^{-\frac{9-\sqrt{33}}{40}t}$$

$$= \frac{66-30\sqrt{33}}{264} e^{-\frac{9+\sqrt{33}}{40}t} + \frac{66+30\sqrt{33}}{264} e^{-\frac{9-\sqrt{33}}{40}t}$$

$$= \left(\frac{11-5\sqrt{33}}{44} e^{-\frac{9+\sqrt{33}}{40}t} + \frac{11+5\sqrt{33}}{44} e^{-\frac{9-\sqrt{33}}{40}t} \right)$$

I apologize for assigning this one!