

c) The two square roots of $re^{i\theta}$ are $\pm \sqrt{r} e^{i\theta/2}$. Find the square roots of the numbers $2-2i\sqrt{3}$ and $-2+2i\sqrt{3}$

$$2-2i\sqrt{3} = 4e^{-\pi/3}$$

$$\theta = \tan^{-1}\left(-\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\text{So, } \sqrt{2-2i\sqrt{3}} = \pm 2e^{-\pi/6} = \{2e^{-\pi/6}, 2e^{i\pi/6}\}$$

$$-2+2i\sqrt{3} = 4e^{2\pi/3}$$

$$\text{So, } \sqrt{-2+2i\sqrt{3}} = \pm 2e^{\pi/3} = \{2e^{\pi/3}, 2e^{-2\pi/3}\}$$

3.4.1

~~Determine the motion~~

Determine the period and frequency of the simple harmonic motion of a 4-kg mass on the end of a spring with spring constant $k=16\text{ N/m}$.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16\text{ N/m}}{4\text{ kg}}} = \boxed{2/\text{s}} = \text{angular frequency}$$

~~$$f = 2\pi\omega = \boxed{4\pi/\text{s}} = \text{frequency}$$~~

~~$$T = 1/f = \boxed{\frac{1}{4\pi}\text{ s}} = \text{period}$$~~

~~$$f = \frac{\omega}{2\pi} = \boxed{\frac{1}{\pi}\text{ /s}} = \text{frequency (cycles per second)}$$~~

$$T = 1/f = \boxed{\pi\text{ s}} = \text{period}$$

52 seconds.

3.4.9.

Assume the DE for the pendulum of length L is $L\theta'' + g\theta = 0$, where $g = GM/R^2$ is the gravitational acceleration at the location of the pendulum.

Two pendulums are of lengths L_1 and L_2 and - when located at the respective distances R_1 and R_2 from the Earth's center - have periods p_1 and p_2 . Show that

$$\frac{p_1}{p_2} = \frac{R_1 \sqrt{L_1}}{R_2 \sqrt{L_2}}$$

We have simple harmonic motion with angular frequency $\omega = \sqrt{g/L}$, and so periods $T = 2\pi/\omega$.

$$p_1 = 2\pi / \sqrt{g/L_1} = 2\pi / \sqrt{\frac{GM}{R_1^2 L_1}}$$

Similarly,

$$p_2 = 2\pi / \sqrt{\frac{GM}{R_2^2 L_2}}$$

So,

$$\frac{p_1}{p_2} = \frac{2\pi / \sqrt{\frac{GM}{R_1^2 L_1}}}{2\pi / \sqrt{\frac{GM}{R_2^2 L_2}}} = \boxed{\frac{R_1 \sqrt{L_1}}{R_2 \sqrt{L_2}}} \quad \checkmark$$

4.18.

Solve the ODE, with both damping and the damping term removed, and graph the two solutions

$$m=2 \quad c=12 \quad k=50 \quad x_0=0 \quad v_0=-8$$

$$(12)^2 - 4(2)(50) = 144 - 400 = -256 < 0$$

So, underdamped.

Roots of

$$2r^2 + 12r + 50 = 0 \quad \text{are}$$

$$r = \frac{-12 \pm \sqrt{-256}}{2(2)} = -3 \pm 4i$$

So, the solution is:

$$x(t) = e^{-3t} (c_1 \cos(4t) + c_2 \sin(4t))$$

$$x(0) = c_1 = 0$$

$$x'(t) = e^{-3t} (-4c_1 \sin(4t) + 4c_2 \cos(4t)) - 3e^{-3t} (c_1 \cos(4t) + c_2 \sin(4t))$$

$$\text{with } c_1 = 0$$

$$x'(t) = 4e^{-3t} c_2 \cos(4t) - 3e^{-3t} c_2 \sin(4t)$$

$$= c_2 e^{-3t} (4 \cos(4t) - 3 \sin(4t))$$

$$x'(0) = 4c_2 = -8 \Rightarrow c_2 = -2$$

So,

$$x'(t) = e^{-3t} (6 \sin(4t) - 8 \cos(4t))$$

We can rewrite this as:

$$x(t) = 10 e^{-3t} \left(\frac{3}{5} \sin(4t) - \frac{4}{5} \cos(4t) \right)$$

$$\alpha = \tan^{-1} \left(\frac{3}{-4} \right) =$$

which gives us

$$x(t) = 10 e^{-3t} \cos(4t -$$

$$\begin{aligned} x(t) &= -2 e^{-3t} \sin(4t) \\ &= -2 e^{-3t} \cos(4t - \frac{\pi}{2}) \end{aligned}$$

Now, without damping we would get

$$\omega = \sqrt{\frac{90}{2}} = 5$$

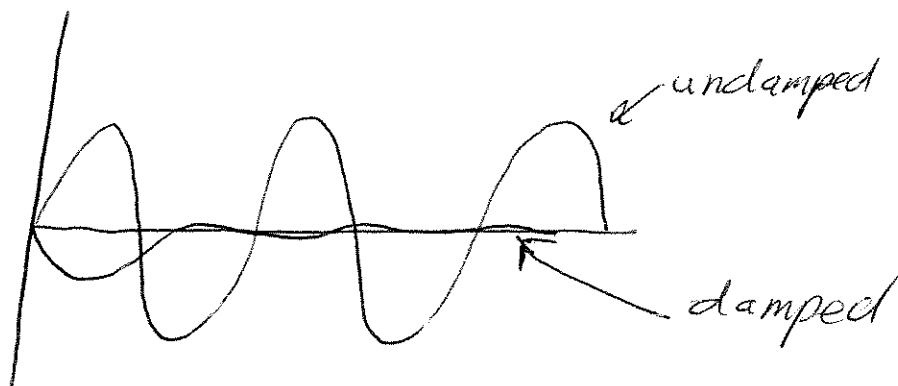
$$x(t) = c_1 \cos(5t) + c_2 \sin(5t)$$

$$c_1 = 0$$

$$x'(t) = 5c_2 \cos(5t) \quad c_2 = -\frac{8}{5}$$

$$x(t) = -\frac{8}{5} \sin(5t) = -\frac{8}{5} \cos(5t - \frac{\pi}{2})$$

Graph



3.4.21

$$m=1 \quad c=10 \quad k=125 \quad x_0=6 \quad v_0=90$$

$$1x'' + 10x' + 125x = 0$$

Solve characteristic equation:

$$r^2 + 10r + 125 = 0$$

$$r = \frac{-10 \pm \sqrt{10^2 - 4(1)(125)}}{2}$$

$$= \frac{-10 \pm \sqrt{100 - 500}}{2}$$

$$= -5 \pm 10i$$

So, our solution is:

$$x(t) = e^{-5t} (c_1 \cos(10t) + c_2 \sin(10t))$$

$$x'(t) = e^{-5t} (-10c_1 \sin(10t) + 10c_2 \cos(10t)) - 5e^{-5t} (c_1 \cos(10t) + c_2 \sin(10t))$$

$$x(0) = c_1 = 6$$

$$x'(0) = 10c_2 - 30 = 90 \Rightarrow c_2 = 8$$

therefore

$$x(t) = e^{-5t} (6 \cos(10t) + 8 \sin(10t))$$

We can rewrite this as:

$$x(t) = 10 e^{-st} \left(\frac{3}{5} \cos(10t) + \frac{4}{5} \sin(10t) \right)$$

$$\cancel{\tan^{-1}(\alpha)} = \boxed{\alpha = \tan^{-1}\left(\frac{4}{3}\right)}$$

$$\boxed{x(t) = 10 e^{-st} \cos(10t - \alpha)}$$

Undamped case:

$$\omega = \sqrt{\frac{125}{1}} = 5\sqrt{5}$$

$$x(t) = c_1 \cos(5\sqrt{5}t) + c_2 \sin(5\sqrt{5}t)$$

$$x(0) = 6 = c_1$$

$$x'(0) = \cancel{30} \cdot 5\sqrt{5} c_2 = 50 \Rightarrow c_2 = 2\sqrt{5}$$

So,

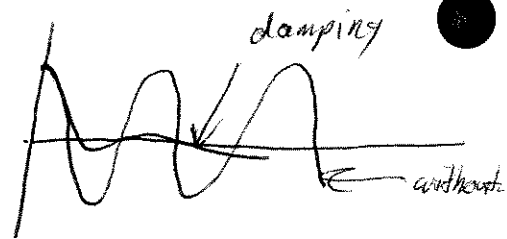
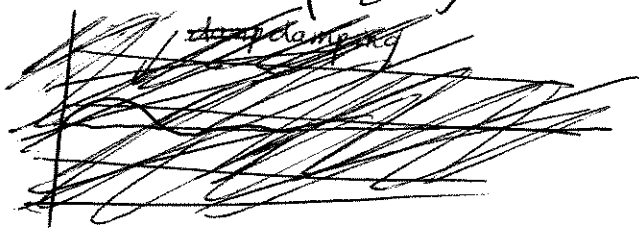
$$x(t) = 6 \cos(5\sqrt{5}t) + 2\sqrt{5} \sin(5\sqrt{5}t)$$

$$\sqrt{6^2 + (2\sqrt{5})^2} = \sqrt{36 + 20} = \sqrt{56} = 2\sqrt{14}$$

$$x(t) = 2\sqrt{14} \left(\frac{6}{2\sqrt{14}} \cos(5\sqrt{5}t) + \frac{2\sqrt{5}}{2\sqrt{14}} \sin(5\sqrt{5}t) \right)$$
$$= 2\sqrt{14} \cos(5\sqrt{5}t - \alpha) \quad \alpha = \tan^{-1}\left(\frac{6}{2\sqrt{5}}\right)$$

$$\alpha = \tan^{-1}\left(\frac{2\sqrt{5}}{6}\right)$$

Graph



3.5.1

Find a particular solution y_p to the ODE:

$$y'' + 16y = e^{3x}$$

$$y_p = Ae^{3x} \quad y_p'' = 9Ae^{3x}$$

$$\Rightarrow \cancel{9A} + 16A (9A + 16A)e^{3x} = e^{3x}$$

$$\Rightarrow A = \frac{1}{25}$$

So,

$$\boxed{y_p = \frac{1}{25} e^{3x}}$$

3.5.11

$$y^{(3)} + 4y' = 3x - 1$$

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p^{(3)} = 0$$

$$\Rightarrow \cancel{2A} \cdot 0 + 4(2Ax + B) = 3x - 1$$

$$8A = 3 \Rightarrow A = \frac{3}{8}$$

$$4B = -1 \Rightarrow B = -\frac{1}{4}$$

C could be anything
Take $C = 0$

$$\boxed{y_p = \frac{3}{8}x^2 - \frac{1}{4}x}$$

3.5.23

Determine the appropriate form of y_p , but not its coefficients.

$$y'' + 4y = 3x \cos(2x)$$

$$r^2 + 4 = 0 \quad \Rightarrow r = \pm 2i$$

So, homogenous solution is:

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

first guess for y_p is:

$$y_p = (Ax + B) \cos(2x) + (Cx + D) \sin(2x)$$

but, $B \cos(2x) + D \sin(2x)$ is not LI from y_h . So, we use:

$$y_p = (Ax^2 + Bx) \cos(2x) + (Cx^2 + Dx) \sin(2x)$$

3.5.28

$$y^{(4)} + 9y'' = (x^2 + 1) \sin 3x$$

characteristic equation:

$$r^4 + 9r^2 = r^2(r + 3i)(r - 3i)$$

has roots $r = 0, 3i, -3i$ with 0 repeated.
So,

$$y_h = c_1 + c_2 x + c_3 \sin(3x) + c_4 \cos(3x)$$

Our first guess for y_p would be:

$$y_p = (Ax^2 + Bx + C) \sin(3x) + (Dx^2 + Ex + F) \cos(3x)$$

but, that's not LI from y_h (the $(C \sin(3x) + F \cos(3x))$ term)

So,

$$y_p = (Ax^3 + Bx^2 + Cx) \sin(3x) + (Dx^3 + Ex^2 + Fx) \cos(3x)$$

3.5.35

Solve the initial value problem.

$$y'' - 2y' + 2y = x + 1 \quad y(0) = 3, \quad y'(0) = 0$$

Solve the characteristic equation:

$$r^2 - 2r + 2 = 0 = (r-1)^2 + 1 = 0$$

$$\Rightarrow r = 1 \pm i$$

So, we get:

$$y_h = e^x (c_1 \cos(x) + c_2 \sin(x))$$

Now, the guess for y_p would be:

$$y_p = Ax + B$$

which has each term LI from each term of y_h .

So,

$$\begin{aligned} y_p &= Ax + B \\ y_p' &= A \\ y_p'' &= 0 \end{aligned}$$

Plugging in we get:

$$-2A + 2(Ax + B) = x + 1$$

$$\Rightarrow 2A = 1 \quad A = \frac{1}{2}$$

$$-2A + 2B = 1 \quad B = 1$$

and so =

$$\begin{aligned}y(x) &= y_h + y_p \\ &= e^x(c_1 \cos(x) + c_2 \sin(x)) + \frac{1}{2}x + 1\end{aligned}$$

Using our initial conditions

$$y(0) = c_1 + 1 = 3 \Rightarrow c_1 = 2$$

$$y'(x) = e^x(2\cos(x) + c_2 \sin(x)) + e^x(-2\sin(x) + c_2 \cos(x)) + \frac{1}{2}$$

$$y'(0) = 2 + c_2 + \frac{1}{2} = 0 \Rightarrow c_2 = -\frac{5}{2}$$

So,

$$y(x) = e^x \left(2\cos(x) - \frac{5}{2}\sin(x) + \frac{1}{2}x + 1 \right)$$

3. 5.47

Use the method of variation of parameters to solve the ODE:

$$y'' + 3y' + 2y = 4e^x$$

Solve for y_h

$$r^2 + 3r + 2 = (r+1)(r+2) \quad r = -1, -2$$

So,

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$$

Now, $y_1 = e^{-x}$ $y_2 = e^{-2x}$

$$W(x) = W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$\int \frac{y_2(x) f(x)}{W(x)} = \int \frac{e^{-2x} (4e^x)}{-e^{-3x}} \\ = - \int 4e^{2x} = -2e^{2x}$$

$$\int \frac{y_1(x) f(x)}{W(x)} = \int \frac{e^{-x} (4e^x)}{-e^{-3x}} \\ = - \int 4e^{3x} = -\frac{4}{3} e^{3x}$$

So,

$$y_p = -e^{-x} (-2e^{2x}) + e^{-2x} \left(-\frac{4}{3} e^{3x} \right) \\ = 2e^x - \frac{4}{3} e^x = \boxed{\frac{2}{3} e^x}$$

Check:

$$y_p'' = \frac{2}{3} e^x \quad y_p' = \frac{2}{3} e^x \quad y_p = \frac{2}{3} e^x$$

$$y_p'' + 3y_p' + 2y_p = \left(\frac{2}{3} + 2 + \frac{4}{3} \right) e^x = \frac{12}{3} e^x = 4e^x \quad \checkmark$$

3.5.56

$$y'' - 4y = xe^x$$

Solving for y_h

~~$$y'' - 4y = 0$$~~
$$r^2 - 4 = 0 \quad r = \pm 2$$

$$y_1 = e^{2x} \quad y_2 = e^{-2x}$$

$$\int W(x) = W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}$$

$$= -2 - 2 = -4$$

$$\int \frac{y_2 f}{W} = \int \frac{e^{-2x} x e^x}{-4} = -\frac{1}{4} \int x e^{-x}$$

$$= \frac{1}{4} x e^{-x} + \frac{1}{4} e^{-x}$$

$$\int \frac{y_1 f}{W} = \int \frac{e^{2x} x e^x}{-4} = -\frac{1}{4} \int x e^{3x}$$

$$= -\frac{1}{12} x e^{3x} + \frac{1}{36} e^{3x}$$

So,

$$y_p = -e^{2x} \left(\frac{1}{4} x e^{-x} + \frac{1}{4} e^{-x} \right) + e^{-2x} \left(-\frac{1}{12} x e^{3x} + \frac{1}{36} e^{3x} \right)$$

$$= -\frac{1}{4} x e^x - \frac{1}{4} e^x - \frac{1}{12} x e^x + \frac{1}{36} e^x$$

$$= \boxed{-\frac{1}{3} x e^x - \frac{2}{9} e^x}$$