3.1.1. Verify y_1 and y_2 are solutions, and then find constants $c_{1,2}$ such that $y = c_1 y_1 + c_2 y_2$ satisfies the given initial conditions

$y'' - y = 0 \qquad y_1 = e^x \quad y_2 = e^{-x} \quad y(0) = 0$
$y'(0) = 5$

$y_1' = e^x$
$y_1'' = e^x$

$y_1'' - y_1 = e^x - e^x = 0$

$y_2' = -e^{-x}$
$y_2'' = e^{-x}$

$y_2'' - y_2 = e^{-x} - e^{-x} = 0$

So, both y_1 and y_2 work.

$y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{-x}$
$y' = c_1 e^x - c_2 e^{-x}$

So, $\quad y(0) = c_1 + c_2 = 0 \quad\Rightarrow c_1 = \frac{5}{2} \quad c_2 = -\frac{5}{2}$
$y'(0) = c_1 - c_2 = 5$

So, $\boxed{y(x) = \frac{1}{2}(5e^x - 5e^{-x})}$

3.1.16. $x^2y'' + xy' + y = 0$ $\quad y_1 = \cos(\ln x)$

$y_2 = \sin(\ln x)$

$y(1) = 2 \qquad y'(1) = 3$

$$y_1' = \frac{-\sin(\ln x)}{x} \qquad y_1'' = \frac{-\cos(\ln x) + \sin(\ln x)}{x^2}$$

$$x^2 y_1'' + x y_1' + y_1 = -\cos(\ln x) + \sin(\ln x) - \sin(\ln x) + \cos(\ln x) = 0$$

So, checks out for y_1.

$$y_2' = \frac{\cos(\ln x)}{x} \qquad y_2'' = \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}$$

$$x^2 y_2'' + x y_2' + y_2 = -\sin(\ln x) - \cos(\ln x) + \cos(\ln x) + \sin(\ln x) = 0$$

So, checks out for y_2.

$$y = c_1 y_1 + c_2 y_2 = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

$$y(1) = c_1 \cos(\ln(1)) + c_2 \sin(\ln(1)) = c_1 = 2$$

$$y'(x) = \frac{-2\sin(\ln x)}{x} + \frac{c_2 \cos(\ln x)}{x}$$

$$y'(1) = \frac{-2\sin(\ln 1)}{1} + \frac{c_2 \cos(\ln 1)}{1} = c_2 = 3$$

$$\Rightarrow \boxed{y(x) = 2\cos(\ln(x)) + 3\sin(\ln(x))}$$

3.1.18 Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that if $c^2 \neq 1$, then $y = cx^3$ is not a solution.

Suppose $y = cx^3$ then $y' = 3cx^2$ and $y'' = 6cx$

So,

$$yy'' = 6c^2x^4 = 6x^4$$

if and only if $c^2 = 1$, and so $c = \pm 1$.

So,

$$y = x^3 \quad \text{or} \quad y = -x^3$$

work, but no other c works.

3.1.24. Determine if $f(x)$ and $g(x)$ are linearly independent on the real line $\mathbb{R}$.

$f(x) = \sin^2 x \qquad g(x) = 1 - \cos 2x$

Now, $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ (basic trig. identity)

So, $2g(x) = f(x)$

and they are not linearly independent

Other proof:

$$W(f,g) = \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cos x & 2\sin 2x \end{vmatrix} = 2\sin^2 x \sin 2x - 2\sin x \cos x (1 - \cos 2x) = 0 \text{ when } x = 0.$$

3.1.39 Find a general solution to the ODE:

$$4y'' + 4y' + y = 0$$

Characteristic Equation:

$$4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2$$

So, $r = -\frac{1}{2}$ is a root of order 2.

So, the general solution will be:

$$\boxed{y(x) = c_1 e^{-x/2} + c_2 x e^{-x/2}}$$

.2.1 Show that the following functions are linearly dependent on $\mathbb{R}$.

$f(x) = 2x$, $g(x) = 3x^2$, $h(x) = 5x - 8x^2$

$$6h(x) + 16g(x) - 15f(x)$$

$$= 6(5x - 8x^2) + 16(3x^2) - 15(2x)$$

$$= 30x - 48x^2 + 48x^2 - 30x$$

$$= \boxed{0}$$

3.2.10 Use the Wronskian to prove the following functions are linearly independent

$$f(x) = e^x \quad g(x) = x^{-2} \quad h(x) = x^{-2}\ln x \quad x > 0$$

$$W(f,g,h) = \begin{vmatrix} e^x & x^{-2} & x^{-2}\ln x \\ e^x & -2x^{-3} & x^{-3} - 2x^{-3}\ln x \\ e^x & 6x^{-4} & -3x^{-4} - 2x^{-4} + 6x^{-4}\ln x \end{vmatrix}$$

$$\cancel{= e^x \begin{vmatrix} -\frac{2}{x^2} & \frac{1}{x^3} - \frac{2\ln x}{x^3} \\ \frac{6}{x^4} & \frac{-5+6\ln x}{x^4} \end{vmatrix}}$$

Evaluate at $x = 1$ to get

$$W(f,g,h)(1) = \begin{vmatrix} e & 1 & 0 \\ e & -2 & 1 \\ e & 6 & -5 \end{vmatrix}$$

$$= e(10-6) - e(-5-0) + e(1-0)$$

$$= 4e + 5e + e = 10e \neq 0$$

So, linearly independent

3.2-16 Find a particular solution to the given ODE with given linearly independent homogenous solutions that satisfy the given initial conditions

$$y^{(3)} - 5y'' + 8y' - 4y = 0; \quad y(0) = 1 \quad y'(0) = 4 \quad y''(0) = 0;$$

$$y_1 = e^x \quad y_2 = e^{2x} \quad y_3 = xe^{2x}$$

$$y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$$

$$y(0) = c_1 + c_2 \cancel{+ c_3}$$

$$y'(x) = c_1 e^x + 2c_2 e^{2x} + 2c_3 x e^{2x} + c_3 e^{2x}$$

$$y'(0) = c_1 + 2c_2 + c_3$$

$$y''(x) = c_1 e^x + 4c_2 e^{2x} + 4c_3 x e^{2x} + 4c_3 e^{2x}$$

$$y''(0) = c_1 + 4c_2 + 4c_3$$

$$\Rightarrow \begin{array}{l} c_1 + c_2 = 1 \\ c_1 + 2c_2 + c_3 = 4 \\ c_1 + 4c_2 + 4c_3 = 0 \end{array} \Rightarrow \begin{array}{l} c_2 + c_3 = 3 \\ 3c_2 + 4c_3 = -1 \end{array} \qquad c_2 = 3 - c_3$$

$$3(3 - c_3) + 4c_3 = -1 \Rightarrow 9 + c_3 = -1 \Rightarrow c_3 = -10.$$

$$\Rightarrow c_2 = 13 \qquad c_1 = -12$$

So,

$$\boxed{y(x) = -12e^x + 13e^{2x} - 10xe^{2x}}$$

3-2-24 Find a solution to the given ODE that satisfies the initial conditions with the given y_c and y_p

$$y''-2y'+2y=2x \quad y(0)=4 \quad y'(0)=8$$

$$y_c = c_1 e^x \cos x + c_2 e^x \sin x \qquad y_p = x+1$$

$$y = c_1 e^x \cos x + c_2 e^x \sin x + x + 1$$

$$y' = -c_1 e^x \sin x + c_1 e^x \cos x + c_2 e^x \cos x + c_2 e^x \sin x + 1$$
$$= (-c_1 + c_2) e^x \sin x + (c_1 + c_2) e^x \cos x + 1$$

$$y(0) = c_1 + 1 = 4 \Rightarrow c_1 = 3$$
$$y'(0) = c_1 + c_2 + 1 = 8 \Rightarrow 3 + c_2 + 1 = 8 \Rightarrow c_2 = 4$$

So,

$$\boxed{y(x) = 3e^x \cos x + 4e^x \sin x + x + 1}$$

3.2.31 This problem indicates why we can only impose n initial conditions on the solution of an n-th order linear differential equation.

a) Given the equation

$$y'' + py' + qy = 0$$

explain why the value of $y''(a)$ is determined by the values of $y(a)$ and $y'(a)$.

Well,

$$y''(a) = -p(a)y'(a) - q(a)y(a).$$

So, if $y(a)$, $p(a)$, $y'(a)$, and $q(a)$ are known then $y''(a)$ is determined.

b) Prove that the equation

$$y'' - 2y' - 5y = 0$$

has a solution satisfying the conditions

$$y(0) = 1 \quad y'(0) = 0 \quad \text{and} \quad y''(0) = C$$

if and only if $C = 5$.

Well,

$$y''(0) = 2y'(0) + 5y(0) = 2(0) + 5(1) = 5$$

So, we must have $y''(0) = 5$.

Dylan Zwick
Math 2280

Assignment #5 Solutions

3.3.1 Find the general solutions of the DE.

$$y'' - 4y = 0$$

characteristic equation

$$r^2 - 4 = 0 \Rightarrow r = \pm 2$$

So,

$$\boxed{y(x) = c_1 e^{-2x} + c_2 e^{2x}}$$

3.3.10 $5y^{(4)} + 3y^{(3)} = 0$

characteristic equation

$$5r^4 + 3r^3 = 0$$

$$\Rightarrow (5r+3)r^3 \qquad r = \{0, -\tfrac{3}{5}\}$$

with 0 a root of order 3. So,

$$y(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 x^2 e^{0x} + c_4 e^{-3/5 x}$$

$$= \boxed{c_1 + c_2 x + c_3 x^2 + c_4 e^{-3/5 x}}$$

3.3.25 Find the solution to the initial value problem

$3y^{(3)} + 2y'' = 0; \quad y(0) = -1, \ y'(0) = 0, \ y''(0) = 1$

characteristic equation:

$3r^3 + 2r^2 = 0 \Rightarrow (3r+2)r^2 = 0$

So, $r = -\frac{2}{3}, 0$ $\quad$ 0 is a 2nd order root.

$y(x) = c_1 + c_2 x + c_3 e^{-2/3 x}$

$y(0) = c_1 + c_3 = -1$

$y'(x) = c_2 - \frac{2}{3} c_3 e^{-2/3 x}$

$y'(0) = c_2 - \frac{2}{3} c_3 = 0$

$y''(x) = \frac{4}{9} c_3 e^{-2/3 x}$

$y''(0) = \frac{4}{9} c_3 = 1$

So,

$c_3 = \frac{9}{4} \quad c_2 = \frac{3}{2} \quad c_1 = -\frac{5}{4}$

$$\boxed{y(x) = -\frac{5}{4} + \frac{3}{2} x + \frac{9}{4} e^{-\frac{2}{3} x}}$$

3.3.30 Find general solutions for the ODE

$y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$

Characteristic Equation

$r^4 - r^3 + r^2 - 3r - 6 = 0$

$(-1)^4 - (-1)^3 + (-1)^2 - 3(-1) - 6 = 1 + 1 + 1 + 3 - 6 = 0$

So $r = -1$ is a root:

$(r+1)(r^3 - 2r^2 + 3r - 6)$

$r = 2$ works as a root for the second equation

$(r+1)(r-2)(r^2 + 3)$

$r = \pm\sqrt{3}\,i$ are the remaining 2 solutions

So, the general solution is:

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos(\sqrt{3}x) + c_4 \sin(\sqrt{3}x)}$$

3.3.43 a) Use Euler's formula to show that every complex number can be written in the form $re^{i\theta}$, where $r \geq 0$ and $-\pi < \theta \leq \pi$.

$$z = x + iy$$

can be written as:

$$z = re^{i\theta} \quad \text{where}$$

$$r = \sqrt{x^2+y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

as $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$ $\quad \sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

for this choice of θ (taking $-\pi < \theta \leq \pi$) and so

$$re^{i\theta} = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{iy}{\sqrt{x^2+y^2}}\right) = x+iy = z$$

using $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula)

b) ~~##~~ Express the numbers $4, -2, 3i, 1+i$, and $-1+i\sqrt{3}$ in the form $re^{i\theta}$.

$4 = 4e^{i0}$ $\quad \theta = 0 \quad r = 4$

$-2 = 2e^{i\pi}$ $\quad \theta = \pi \quad r = 2$

$3i = 3e^{i\pi/2}$ $\quad \theta = \pi/2 \quad r = 3$

$1+i = \sqrt{2}e^{i\pi/4}$ $\quad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} \quad r = \sqrt{1^2+1^2} = \sqrt{2}$

~~$-1+i\sqrt{3} = 2e^{+\pi/3}$ $\quad \theta = \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3} \quad r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$~~

$-1+i\sqrt{3} = 2e^{2\pi/3}$ $\quad \theta = \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) = \frac{2\pi}{3} \quad r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$

c) The two square roots of $re^{i\theta}$ are $\pm\sqrt{r}\,e^{i\theta/2}$. Find the square roots of the numbers $2-2i\sqrt{3}$ and $-2+2i\sqrt{3}$

$2-2i\sqrt{3} = 4e^{-\pi/3}$ $\qquad \theta = \tan^{-1}\left(-\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

$r = \sqrt{2^2+(-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

So, $\sqrt{2-2i\sqrt{3}} = \pm 2e^{-\pi/6} = \{2e^{-\pi/6}, 2e^{5\pi/6}\}$

$-2+2i\sqrt{3} = 4e^{2\pi/3}$

So, $\sqrt{-2+2i\sqrt{3}} = \pm 2e^{\pi/3} = \{2e^{\pi/3}, 2e^{-2\pi/3}\}$

3.4.1 ~~Determine the motion~~

Determine the period and frequency of the simple harmonic motion of a 4-kg mass on the end of a spring with spring constant $k = 16 N/m$.

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16\ N/m}{4kg}} = \boxed{2/s}$ = angular frequency

~~$f = 2\pi\omega = \boxed{4\pi/s}$ = frequency~~

~~$T = 1/f = \boxed{\frac{1}{4\pi}s}$ = period~~

~~$f = \frac{\omega}{2\pi} = \boxed{\frac{1}{\pi}/s}$ = frequency (cycles per second)~~

$T = 1/f = \boxed{\pi\ s}$ = period

s = seconds.