Dylan Zwick

Assignment # 3 Solutions

2.1.1. $\frac{dx}{dt} = x - x^2 \qquad x(0) = 2$

$$\Rightarrow \frac{dx}{x - x^2} = dt$$

$$\Rightarrow \int \left(\frac{A}{x} + \frac{B}{1-x} \right) dx = \int dt$$

$$\Rightarrow A(1-x) + Bx = 1 \Rightarrow A = 1, \quad B = 1$$

$$\int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx = t + C$$

$$\Rightarrow \ln|x| - \ln|1-x| = t + C$$

$$\Rightarrow \ln\left|\frac{x}{1-x}\right| = t + C \Rightarrow \frac{x}{1-x} = Ce^t$$

$$\Rightarrow x = (1-x)Ce^t$$

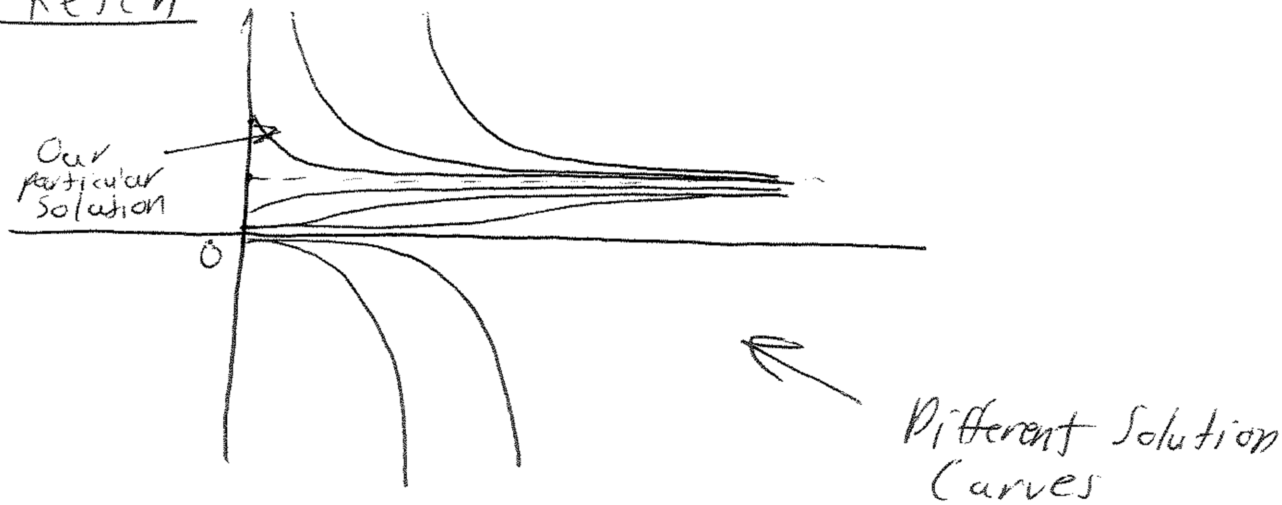
$$\Rightarrow x(1 + Ce^t) = Ce^t \Rightarrow x(t) = \frac{Ce^t}{1 + Ce^t} = \boxed{\frac{1}{1 + Ce^{-t}}}$$

$$x(0) = \frac{1}{1+C} = 2 \Rightarrow C = -\frac{1}{2}$$

So,

$$\boxed{x(t) = \frac{2}{2 - e^{-t}}}$$

Sketch



8.

$$\frac{dx}{dt} = 7x(x-13) \qquad x(0)=17$$

$$\Rightarrow \frac{dx}{7x(x-13)} = dt$$

$$\Rightarrow \int \left(\frac{A}{7x} + \frac{B}{x-13} \right) dx = t + C$$

$$A(x-13) + B(7x) = 1$$

$$A = -\frac{1}{13} \qquad B = \frac{1}{91}$$

$$\Rightarrow \int \left(-\frac{1}{91x} + \frac{1}{91(x-13)} \right) dx = t + C$$

$$-\frac{\ln|x|}{91} + \frac{\ln|x-13|}{91} = t + C$$

$$\Rightarrow \ln\left| \frac{x-13}{x} \right| = 91t + C$$

$$\frac{x-13}{x} = Ce^{91t}$$

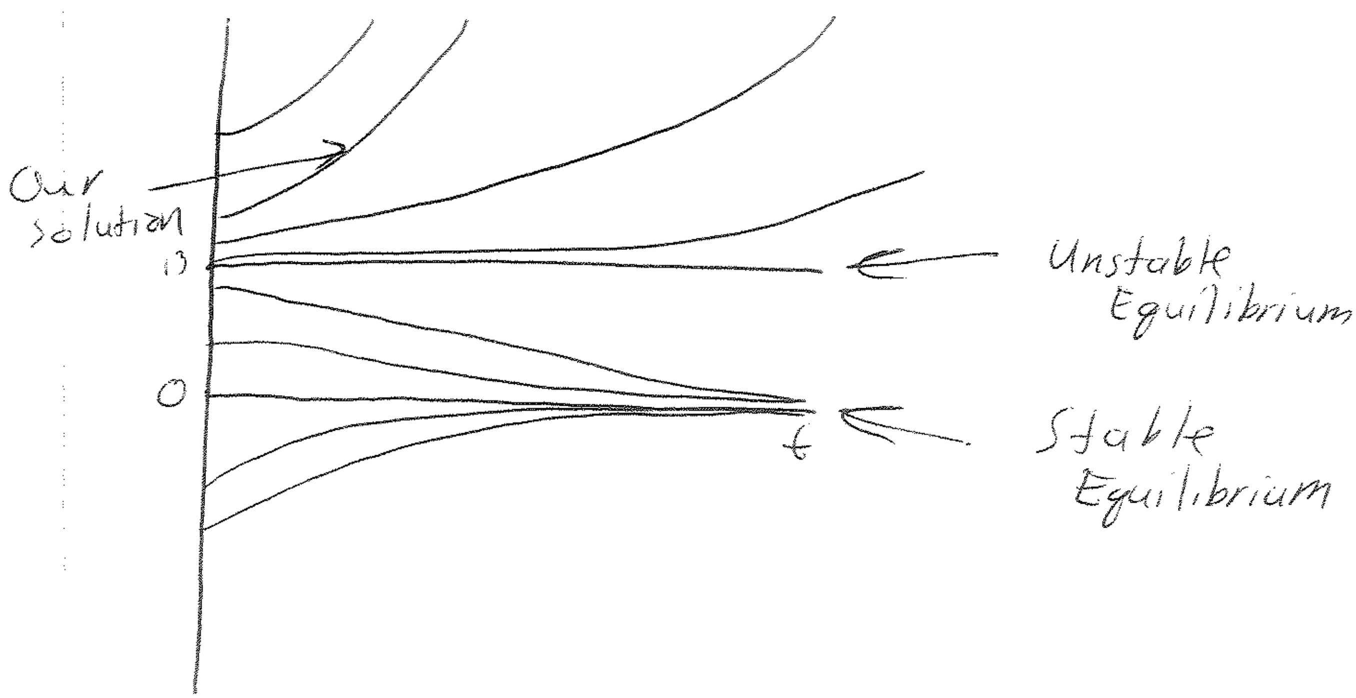
$$\Rightarrow \quad x - 13 = x\,Ce^{91t}$$
$$\Rightarrow \quad x(1 - Ce^{91t}) = 13$$

$$x(t) = \frac{13}{1 - Ce^{91t}} \qquad\qquad x(0) = 17$$

$$\Rightarrow \quad x(0) = \frac{13}{1 - C} = 17 \Rightarrow 13 = 17 - 17C$$
$$\Rightarrow \quad 17C = 4 \qquad C = 4/17$$

$$x(t) = \frac{221}{17 - 4e^{91t}}$$

<u>Sketch</u>



11. Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are inversely proportional to $\sqrt{P}$.

a) Show that

$$P(t) = \left(\tfrac{1}{2} kt + \sqrt{P_0}\right)^2$$

where k is a constant.

Note: Should say <u>directly</u> proportional to $\sqrt{P}$. Inversely doesn't work!

$$\frac{dP}{dt} = (\beta - \gamma) = k_1\sqrt{P} - k_2\sqrt{P}$$

$$k = k_1 - k_2$$

$$\Rightarrow \frac{dP}{dt} = k\sqrt{P}$$

$$\Rightarrow \frac{dP}{\sqrt{P}} = k\,dt$$

$$\Rightarrow 2\sqrt{P} = kt + C$$

$$\Rightarrow P(t) = \left(\tfrac{1}{2} kt + C\right)^2$$

$$P(0) = P_0 = C^2 \Rightarrow C = \sqrt{P_0}$$

So,

$$\boxed{P(t) = \left(\tfrac{1}{2} kt + \sqrt{P_0}\right)^2}$$

b) If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many fish will there be after 1 year?

$$P(t) = \left(\tfrac{1}{2}kt + \sqrt{100}\right)^2$$

$$\Rightarrow P(6) = \left(\tfrac{1}{2}k(6) + 10\right)^2 = 169$$

$$\Rightarrow \tfrac{1}{2}k(6) + 10 = 13$$

$$\Rightarrow 3k = 3 \quad \Rightarrow k = 1$$

So,

$$P(t) = \left(\tfrac{1}{2}t + 10\right)^2$$

$$P(12) = \left(\tfrac{1}{2}(12) + 10\right)^2 = 16^2 = \boxed{256}$$

16. Consider a rabbit population $P(t)$ satisfying the logistic equation as in problem 15. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occuring at time $t=0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

$$\frac{dP}{dt} = aP - bP^2$$

$P(0) = 120$

$a\,P(0) = 8$

$\Rightarrow a = 8/120 = \frac{2}{30} = \frac{1}{15}$

$b\,P(0)^2 = 6$

$b\,(120)^2 = 6 \Rightarrow b = \frac{1}{2400}$

So,

$$\frac{dP}{dt} = \frac{1}{15}P - \frac{1}{2400}P^2$$

Solving for $P(t)$ we get.

$$\frac{dP}{dt} = \frac{1}{15}P - \frac{1}{2400}P^2$$

$$\int 2400\left(\frac{dP}{160P - P^2}\right) = \int dt$$

$$2400 \int \frac{A}{P} + \frac{B}{160-P} = t + C$$

$$A(160-P) + BP = 1 \qquad A = \frac{1}{160} \quad B = \frac{1}{160}$$

$$\Rightarrow 15 \int \frac{1}{P} + \frac{1}{160-P} = t + C$$

$$\Rightarrow 15\left(\ln|P| - \ln|160-P|\right) = t + C$$

$$\Rightarrow \ln\left|\frac{P}{160-P}\right| = \frac{t}{15} + C \Rightarrow \frac{P}{160-P} = Ce^{t/15}$$

$$\Rightarrow P(1 + Ce^{t/15}) = 160Ce^{t/15}$$

$$\Rightarrow P(t) = \frac{160Ce^{t/15}}{1 + Ce^{t/15}} = \frac{160}{1 + Ce^{-t/15}}$$

$$P(0) = 120 = \frac{160}{1+C} \qquad C = \frac{40}{120} = \frac{1}{3}$$

$$P(t) = \frac{480}{3 + e^{-t/15}} \qquad M = 160$$

$$95\% \text{ of } M = 152$$

$$152 = \frac{480}{3 + e^{-t/15}} \Rightarrow e^{-t/15} = \frac{480}{152} - 3$$

$$\Rightarrow t = -15\ln\left(\frac{480}{152} - 3\right) = \boxed{27.69}$$

29.

During the period from 1790 to 1930, the U.S. population $P(t)$ (t in years) grew from 3.9 million to 123.2 million. Throughout this period, $P(t)$ remained close to the solution of the initial value problem

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2 \quad P(0) = 3.9$$

a) What 1930 population does this logistic equation predict?

$$\frac{dP}{dt} = 0.0001489\,P\,(210.54 - P)$$

$$P(t) = \frac{(210.54)(3.9)}{3.9 + (206.64)e^{-.03135t}}$$

$$\boxed{P(140) = 127.0}$$

So, 127 million people.

b) What limiting population does it predict?

$$M = \frac{.03135}{.0001489} = \boxed{210.54}$$

So, about 210.5 million people.

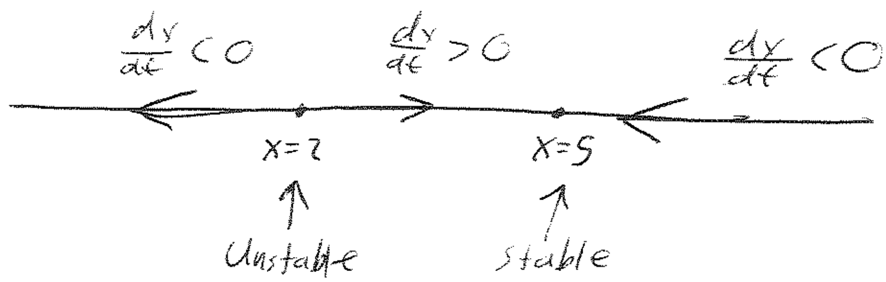
c) Has this logistic equation continued since 1930 to accurately model the U.S population?

$\boxed{\text{No}}$. The current U.S. population is above 300 million.

10.
$$\frac{dx}{dt} = 7x - x^2 - 10$$

$$\frac{dx}{dt} = 7x - x^2 - 10 = -(x-5)(x-2) = (5-x)(x-2)$$

So, equilibrium points where $\frac{dx}{dt} = 0$ are at $x = 5$ and $x = 2$



$$\int \frac{dx}{(5-x)(x-2)} = \int dt$$

$$\int \left(\frac{A}{5-x} + \frac{B}{x-2} \right) dx = t + C$$

$A(x-2) + B(5-x) = 1$
$-2A + 5B = 1$
$A - B = 0 \Rightarrow A = B$
$A = B = \frac{1}{3}$

$$= \frac{1}{3} \int \left(\frac{1}{5-x} + \frac{1}{x-2} \right) dx = t + C$$

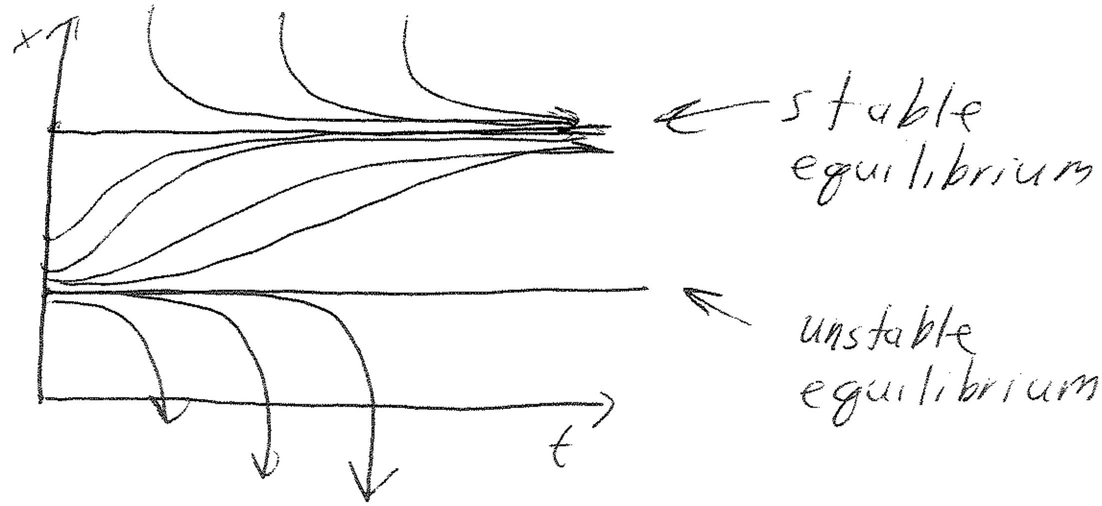
$$\Rightarrow \frac{1}{3} \left(\ln|x-2| - \ln|5-x| \right) = t + C$$

$$\Rightarrow \ln\left| \frac{x-2}{5-x} \right| = 3t + C \Rightarrow \frac{x-2}{5-x} = Ce^{3t}$$

$$\Rightarrow x - 2 = (5-x)Ce^{3t} \Rightarrow x(1 + Ce^{3t}) = 2 + 5Ce^{3t}$$

$$\boxed{x(t) = \frac{2 + 5Ce^{3t}}{1 + Ce^{3t}}}$$

Solutions



21. Consider the differential equation

$$\frac{dx}{dt} = kx - x^3.$$

a) If $k \leq 0$ show that the only critical value $c = 0$ of x is stable.

If $k \leq 0$ then $\frac{dx}{dt}$ has roots

$$kx - x^3 = 0$$

$\Rightarrow \quad kx = x^3 \qquad x = 0$ is a root

$\Rightarrow \quad k = x^2 \qquad$ if $k \leq 0$ this is either 0 or imaginary.

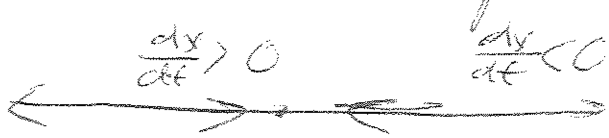
So, only one real root.

At $x = 0$ we have for $x > 0$

$kx - x^3 < 0 \qquad$ for $k \leq 0$, $x > 0$

$kx - x^3 > 0 \qquad$ for $k \leq 0$, $x < 0$

so we have a phase diagram



which we can see is stable at $x = 0$.

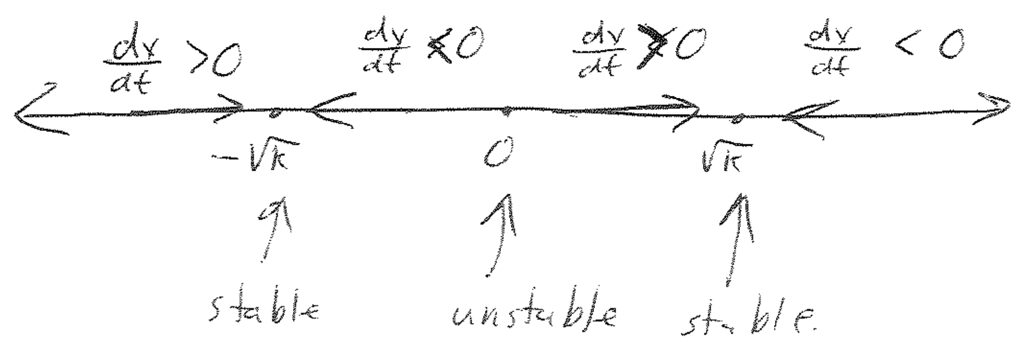
b) If $k>0$, show that the critical point $c=0$ is now unstable, but that the critical points $\pm\sqrt{k}$ are stable. Thus the qualitative nature of the solutions changes at $k=0$ as the parameter k increases; and so $k=0$ is a bifurcation point for the differential equation with parameter k. The plot of all points of the form (k,c) where c is a critical point of the equation $x'=kx-x^3$ is the "pitchfork diagram" shown in Fig. 2.2.13 of the text.

$\frac{dx}{dt} = kx - x^3$ has roots $x=0$ and $\pm\sqrt{k}$

$$kx - x^3 = -x(x+\sqrt{k})(x-\sqrt{k})$$

so, as $k>0$ these are three distinct, real roots

The phase diagram is:



24. Separate variables in the logistic ~~rates~~ harvesting equation:

$$\frac{dx}{dt} = k(N-x)(x-H)$$

and then use partial fractions to derive the solution given in equation (15)

$$\int \frac{dx}{(N-x)(x-H)} = \int k\,dt$$

$$\int \left(\frac{A}{N-x} + \frac{B}{x-H}\right) dx = kt + C$$

$$A(x-H) + B(N-x) = 1$$

$$A - B = 0 \Rightarrow A = B$$

$$-AH + BN = 1 \qquad A(N-H) = 1 \quad A = \frac{1}{N-H} = B$$

$$\Rightarrow \frac{1}{N-H} \int \left(\frac{1}{N-x} + \frac{1}{x-H}\right) dx = kt + C$$

$$\Rightarrow \frac{1}{N-H} \ln\left|\frac{x-H}{N-x}\right| = kt + C$$

$$\Rightarrow \ln\left|\frac{x-H}{N-x}\right| = k(N-H)t + C$$

$$\Rightarrow \frac{x-H}{N-x} = Ce^{k(N-H)t}$$

$$\Rightarrow x - H = (N-x)\,Ce^{k(N-H)t}$$

$$\Rightarrow x\left(1 + Ce^{k(N-H)t}\right) = H + NCe^{k(N-H)t}$$

$$x(t) = \frac{H + NCe^{k(N-H)t}}{1 + Ce^{k(N-H)t}}$$

Now,

$$x(0) = x_0 = \frac{H + NC}{1 + C}$$

$$\Rightarrow \quad x_0 + Cx_0 = H + NC$$

$$\Rightarrow \quad C(x_0 - N) = H - x_0 \qquad C = \frac{H - x_0}{x_0 - N}$$

$$x(t) = \frac{H + N\left(\frac{H - x_0}{x_0 - N}\right) e^{k(N-H)t}}{1 + \left(\frac{H - x_0}{x_0 - N}\right) e^{k(N-H)t}} \times \frac{e^{-k(N-H)t}}{e^{-k(N-H)t}}$$

$$= \frac{H(x_0 - N)\, e^{-k(N-H)t} + N(H - x_0)}{(H - x_0) + (x_0 - N)\, e^{-k(N-H)t}} \times \frac{-1}{-1}$$

$$= \boxed{\frac{N(x_0 - H) - H(x_0 - N)\, e^{-k(N-H)t}}{(x_0 - H) - (x_0 - N)\, e^{-k(N-H)t}}}$$

Equation 2.2.15 from the text, which is what we wanted.

2.3.1. The acceleration of a Maserati is proportional to the difference between 250 km/h and the ~~#~~ velocity of this sports car. If this machine can accelerate from rest to 100 km/h in 10 s, how long will it take for the car to accelerate from rest to 200 km/h.

$$\frac{dv}{dt} = k(250 - v)$$

$$\int \frac{dv}{250 - v} = \int k\,dt$$

$$-\ln(250 - v) = kt + C$$

$$\Rightarrow 250 - v = Ce^{-kt}$$

$$\Rightarrow v(t) = \cancel{Ce^{-kt}}\ 250 - Ce^{-kt}$$

$$v(0) = 0 \Rightarrow C = 250$$

$$\Rightarrow v(t) = 250(1 - e^{-kt})$$

$$v(10) = 100 = 250(1 - e^{-kt})$$

$$\Rightarrow \frac{2}{5} = 1 - e^{-kt} \Rightarrow e^{-kt} = \frac{3}{5} \quad t = 10$$

$$\cancel{\beta} \quad -10k = \ln\left(\frac{3}{5}\right) \Rightarrow k = -\frac{\ln(3) - \ln(5)}{10} = .05108$$

$$\Rightarrow v(t) = 250(1 - e^{.05108t})$$

$$= 200 \Rightarrow \frac{4}{5} = 1 - e^{-.05108t}$$

$$\Rightarrow t = \frac{-\ln(1/5)}{.05108} = \frac{\ln(5)}{.05108} = \boxed{31.5s}$$

4. Consider a body that moves horizontally through a medium whose resistance is proportional to the square of the velocity v, so that $dv/dt = -kv^2$. Show that

$$v(t) = \frac{v_0}{1+v_0kt}$$

and that

$$x(t) = x_0 + \frac{1}{k}\ln(1+v_0kt)$$

Note that in contrast with the result of problem 2, $x(t) \to \infty$ as $t \to \infty$. Which offers less resistance when the body is moving fairly slowly - the medium in this problem or the one in problem 2? Does your answer seem consistent with the observed behaviors of $x(t)$ as $t \to \infty$?

$$\frac{dv}{dt} = -kv^2 \Rightarrow \int \frac{dv}{v^2} = \int -k\,dt$$

$$\Rightarrow -\frac{1}{v} = -kt + C$$

$$\Rightarrow v = \frac{1}{kt+C} \qquad v(0) = \frac{1}{C} \Rightarrow C = \frac{1}{v_0}$$

$$\Rightarrow v(t) = \frac{1}{kt + \frac{1}{v_0}} = \boxed{\frac{v_0}{1+v_0kt}}$$

So,

$$\frac{dx}{dt} = v(t) = \frac{V_0}{1+v_0 k t}$$

$$\Rightarrow \quad x = \int \frac{v_0\,dt}{1+v_0 k t} = \frac{v_0 \ln(1+v_0 k t)}{v_0 k} + C$$

$$= C + \frac{1}{k}\ln(1+v_0 k t)$$

$$x(0) = x_0 = C + \frac{1}{k}\ln(1) = C \Rightarrow C = x_0$$

So,

$$\boxed{x(t) = x_0 + \frac{1}{k}\ln(1+v_0 k t)}$$

For $|v| < 1$ we have $v^2 < |v|$, and so the drag is smaller for fairly small values of v. This is why the distance can go forever, and is not finite.

10. A woman bails out of an airplane at an altitude of 10,000 ft. falls freely for 20 s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance ρv ft/s², taking $\rho = .15$ when the parachute is not open, and $\rho = 1.5$ when the parachute is open.

First, calculate the distance traveled in the first 20 seconds:

$$\frac{dv}{dt} = g - \rho_1 v$$

$$\cancel{\Rightarrow v(t) = v_0 e^{-\rho_1}}$$

$$\int \frac{dv}{g-\rho_1 v} = \int dt$$

$$-\frac{\ln|g-\rho_1 v|}{\rho_1} = t + C$$

$$\Rightarrow \ln(g-\rho_1 v) = -\rho_1 t + C$$

$$\Rightarrow g - \rho_1 v = Ce^{-\rho_1 t}$$

$$\Rightarrow v(t) = \frac{g}{\rho_1} - Ce^{-\rho_1 t}$$

$$v(t)=0 \Rightarrow C = \frac{g}{\rho_1}$$

$$v(t) = \frac{g}{\rho_1}(1-e^{-\rho_1 t})$$

$$v(t) = \frac{dx}{dt} = \frac{g}{\rho_1}(1-e^{-\rho_1 t})$$

$$x(t) = \frac{g}{\rho_1}\left(t + \frac{e^{-\rho_1 t}}{\rho_1}\right) + C$$

$$x(0) = 10{,}000 = \frac{g}{\rho_1^2} + C$$

$$C = 10{,}000 - \frac{g}{\rho_1^2}$$

$$\Rightarrow x(t) = 10{,}000 + \frac{g}{\rho_1}\left(t + \frac{1}{\rho_1}(e^{-\rho_1 t} - 1)\right)$$

If we plug $t = 20$ here we get

$$x(20) = 10,000 - \frac{32.2}{.15}\left(20 + \frac{1}{.15}\left(e^{-.15(20)} - 1\right)\right) \quad g = 32.2\ ft/s^2$$

$$= 7,060.527\ ft$$

Now, we must solve for the time before the rest of the distance is covered

$$0 = 7,060.527 - \frac{32.2}{1.5}\left(t + \frac{1}{1.5}\left(e^{-1.5t} - 1\right)\right)$$

$$\Rightarrow t + \frac{1}{1.5}\left(e^{-1.5t} - 1\right) = 7,060.527 \cancel{\left(\frac{32.2}{1.5}\right)}\left(\frac{1.5}{32.2}\right)$$

$e^{-1.5t} \approx 0$ for even moderately large values of t, so

$$t \approx 7,060.527\left(\frac{1.5}{32.2}\right) + \frac{1}{1.5} \approx 329s$$

So, total time of about $t_{total} = (329 + 20)s$

$$= 349s \approx \boxed{5\ min\ 49s}$$