7.4.1 Find the convolution $f(t) * g(t)$.

$$f(t) = t, \quad g(t) = 1.$$

$$\cancel{f(t) * g(t) = \int_0^t d\tau = \tau \Big|_0^t = t.}$$

$$f(t) * g(t) = \int_0^t \tau \, d\tau = \frac{\tau^2}{2} \Big|_0^t = \frac{t^2}{2} - 0 = \boxed{\frac{t^2}{2}}$$

7.4.5 Find the convolution $f(t) * g(t)$

$$f(t) = g(t) = e^{at}$$

$$f(t) * g(t) = \int_0^t e^{a\tau} e^{a(t-\tau)} d\tau$$

$$= \int_0^t e^{at} d\tau = e^{at} \tau \Big|_0^t = \boxed{t e^{at}}$$

7.4.10. Apply the convolution theorem to find the inverse Laplace transform.

$$F(s) = \frac{1}{s^2(s^2+k^2)} = \frac{1}{s^2}\left(\frac{1}{s^2+k^2}\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+k^2}\right) = \frac{\sin(kt)}{k}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

So,

$$\mathcal{L}^{-1}\left(\left(\frac{1}{s^2}\right)\left(\frac{1}{s^2+k^2}\right)\right) = f(t) * g(t)$$

$$f(t) = t \qquad g(t) = \frac{\sin(kt)}{k}$$

$$f(t) * g(t) = \int_0^t \tau \frac{\sin(k(t-\tau))}{k} d\tau$$

$$u = \tau \qquad dv = \frac{\sin(k(t-\tau))}{k}$$
$$du = d\tau$$
$$v = \frac{\cos(k(t-\tau))}{k^2}$$

$$\Rightarrow f(t) * g(t) = \frac{\tau \cos(k(t-\tau))}{k^2}\Big|_0^t - \int_0^t \frac{\cos(k(t-\tau))}{k^2} d\tau$$

$$= \frac{t}{k^2} + \frac{\sin(k(t-\tau))}{k^3}\Big|_0^t$$

$$= \boxed{\frac{t}{k^2} - \frac{\sin(kt)}{k^3}}$$

Find the Laplace transform

$$f(t) = \frac{\sin t}{t}$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

where $F(\sigma) = \mathcal{L}(\sin t) = \dfrac{1}{\sigma^2+1}$

$$\Rightarrow \mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{d\sigma}{\sigma^2+1}$$

$$= \tan^{-1}(\sigma)\Big|_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \boxed{\frac{\pi}{2} - \tan^{-1}(s)}$$

7.4.31. Find the solution to the given ~~ODE~~ with $x(0)=0$

$$t x'' - (4t+1)x' + 2(2t+1)x = 0$$

$$\cancel{\mathcal{L}\{t x''\} = tX}$$

$$\mathcal{L}\{x''\} = s^2 X(s) - k \quad \text{where } k = x'(0)$$
$$\mathcal{L}\{x'\} = sX(s)$$
$$\mathcal{L}\{x\} = X(s)$$

$$\mathcal{L}\{tx''\} = -\frac{d}{ds}(s^2X(s) - k) = -s^2X'(s) - 2sX(s)$$

$$\mathcal{L}\{-tx'\} = sX'(s) + X(s)$$

$\mathcal{L}\{t x\} = -X'(s).$

So, the ODE becomes:

$$-s^2 X'(s) - 2s X(s) + 4s X'(s) + 4X(s) - s X'(s) - 4X'(s) + 2X(s) = 0$$

$$\Rightarrow (s^2 - 4s + 4) X'(s) + (3s - 6) X(s) = 0$$

$$(s-2)^2 X'(s) + 3(s-2) X(s) \qquad s > 2$$

$$X'(s) + \frac{3}{(s-2)} X(s) = 0.$$

$$p(s) = \int \frac{3}{(s-2)}\, ds = 3\ln(s-2)$$

$$\cancel{e^{p(s)} = \frac{1}{(s-2)^3}}$$

$$\cancel{\Rightarrow \frac{d}{ds}\left(\frac{X(s)}{(s-2)^3}\right) = 0}$$

$$\cancel{\Rightarrow X(s) = C(s-2)^3}$$

$$e^{p(s)} = (s-2)^3$$

$$\Rightarrow \frac{d}{ds}\left(X(s)(s-2)^3\right) = 0$$

$$\Rightarrow X(s) = \frac{C}{(s-2)^3} \Rightarrow x(t) = \boxed{C e^{2t} t^2}$$

Dylan Zwick
Math 2280
Solutions

Assignment #11

7.5.1. Find the inverse Laplace transform and sketch the graph of the inverse transform.

$$F(s) = \frac{e^{-3s}}{s^2}$$

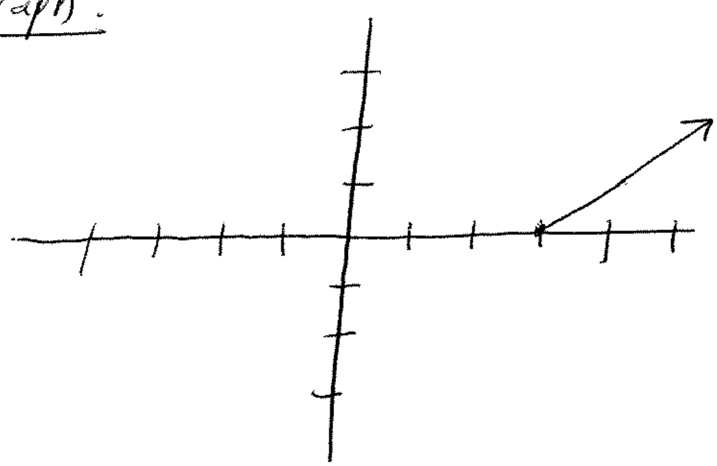
$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{e^{-3s}\tfrac{1}{s^2}\} = u(t-3)f(t-3)$$

where $f(t) = \mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$.

So,

$$f(t) = u(t-3)(t-3)$$

Graph:



7.5.6.

$$F(s) = \frac{s e^{-s}}{s^2+\pi^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left(e^{-s}\left(\frac{s}{s^2+\pi^2}\right)\right)$$

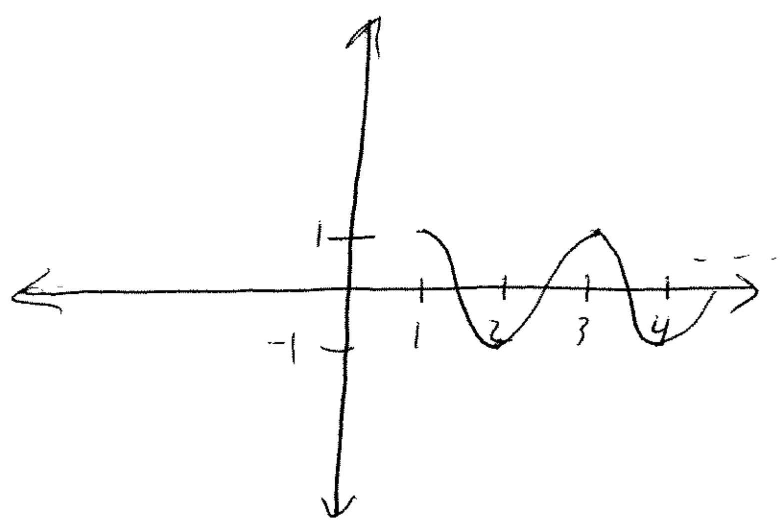
$$= u(t-1)\,f(t-1)$$

where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos(\pi t)$$

So,

$$\mathcal{L}^{-1}\{F(s)\} = u(t-1)\cos(\pi(t-1))$$



?. 5. 15 Find the Laplace transform of the function $f(t)$:

$$f(t) = \sin(t) \text{ if } 0 \le t \le 3\pi$$
$$f(t) = 0 \text{ if } t > 3\pi$$

$$= s(t)\,[u(t) - u(t-3\pi)]$$

$$= \cancel{3\pi}\; u(t)\sin(t) - u(t-3\pi)\sin(t)$$

Now,

$$u(t)\sin(t) - u(t-3\pi)\sin(t)$$

$$= u(t)\sin(t) + u(t-3\pi)\sin(\cancel{3\pi}\, t-3\pi)$$

$$\mathcal{L}\{u(t)\sin(t) + u(t-3\pi)\sin(t-3\pi)\}$$

$$= \frac{1}{s^2+1} + e^{-3\pi s}\left(\frac{1}{s^2+1}\right)$$

$$= \boxed{\frac{1+e^{-3\pi s}}{s^2+1}}$$

7.5.21. $f(t) = t$ if $t \le 1$
$f(t) = 2 - t$ if $1 \le t \le 2$
$f(t) = 0$ if $t > 2$.

$$f(t) = t - u(t-1)t + u(t-1)(2-t) - u(t-2)(2-t)$$

$$= t - u(t-1)(t-1) - u(t-1) + 2u(t-1) - u(t-1)(t-1) - u(t-1) - 2u(t-2) + u(t-2)(t-2) + 2u(t-2)$$

$$= t - u(t-1)(t-1) - u(t-1) + 2u(t-1) - u(t-1)(t-1) - u(t-1) + u(t-2)(t-2)$$

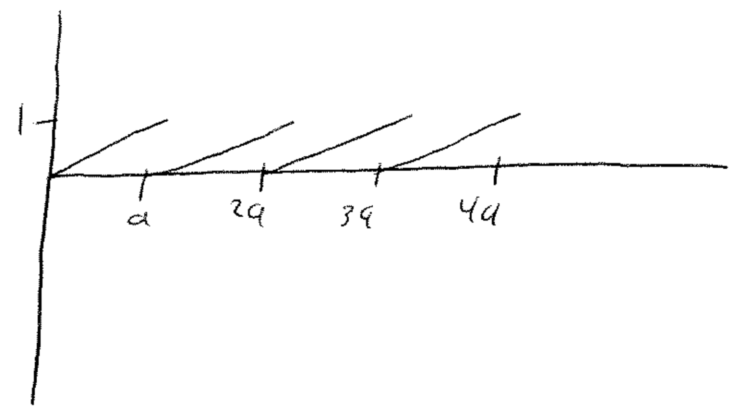
$$= t - 2u(t-1)(t-1) + u(t-2)(t-2)$$

$$\mathcal{L}\{t - 2u(t-1)(t-1) + u(t-2)(t-2)\}$$

$$= \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

$$= \frac{1 - 2e^{-s} + e^{-2s}}{s^2} = \boxed{\frac{(1-e^{-s})^2}{s^2}}$$

7.5.26 Apply Theorem 2 to show that the Laplace transform of the sawtooth function $f(t)$:



is $F(s) = \dfrac{1}{as^2} - \dfrac{e^{-as}}{s(1-e^{-as})}$

The period here is a, and we have, according to Theorem 2:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-as}} \int_0^a \frac{e^{-st}\,t}{a}\,dt$$

$$\int_0^a \frac{e^{-st}t}{a}\,dt \qquad u = t/a \quad du = \tfrac{1}{a}dt \qquad dv = e^{-st} \quad v = -e^{-st}/s$$

$$= -\frac{te^{-st}}{as}\Big|_0^a + \frac{1}{as}\int_0^a e^{-st}\,dt$$

$$= \frac{-ae^{-as}}{as} + \frac{1}{as}\left(\frac{-e^{-st}}{s}\right)\Big|_0^a$$

$$= \frac{-ae^{-as}}{as} - \frac{e^{-as}}{as^2} + \frac{1}{as^2} = \frac{1}{as^2} - \frac{(1+as)e^{-as}}{as^2}$$

So,

$$\mathcal{L}\{f(t)\} = \left(\frac{1}{1-e^{-as}}\right)\left(\frac{1}{as^2} - \frac{(1+as)e^{-as}}{as^2}\right)$$

$$= \left(\frac{1}{1-e^{-as}}\right)\left(\frac{1-e^{-as}}{as^2}\right) - \frac{as\,e^{-as}}{as^2(1-e^{-as})}$$

$$= \frac{1}{as^2} - \frac{a\,e^{-as}}{as(1-e^{-as})}$$

$$= \boxed{\frac{1}{as^2} - \frac{e^{-as}}{s(1-e^{-as})}}$$

7.6.1. Solve the initial value problems:

$$x'' + 4x = \delta(t); \quad x(0) = x'(0) = 0$$

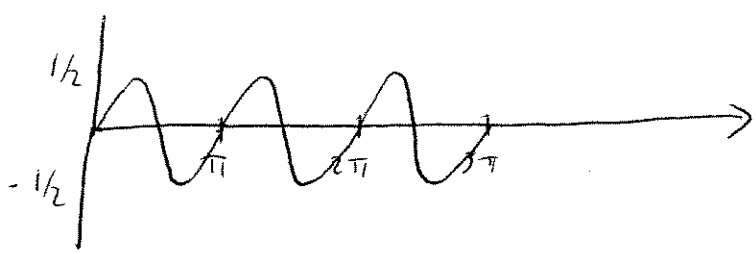
Taking the Laplace transform of both sides:

$$s^2 X(s) + 4X(s) = 1$$

$$\Rightarrow \quad X(s) = \frac{1}{s^2+4}$$

$$\Rightarrow \boxed{x(t) = \tfrac{1}{2}\sin(2t)}$$

Graph:



7.6.6 Solve the initial value problem:

$$x'' + 9x = \delta(t-3\pi) + \cos(3t); \quad x(0) = x'(0) = 0.$$

Taking the Laplace transform:

$$s^2 X(s) + 9X(s) = e^{-3\pi s} + \frac{s}{s^2+9}$$

$$X(s) = \frac{e^{-3\pi s}}{s^2+9} + \frac{s}{(s^2+9)^2}$$

The inverse Laplace transform of this is:

$$\mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{s^2+9}\right) = u(t-3\pi)\, f(t-3\pi)$$

where

$$f(t) = \mathcal{L}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$$

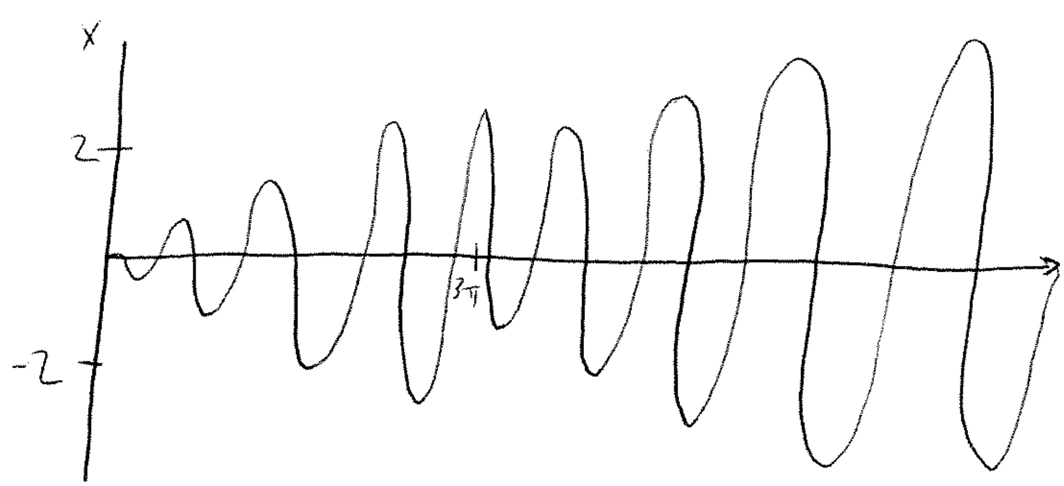
$$f(t-3\pi) = \frac{1}{3}\sin(3(t-3\pi)) = -\frac{1}{3}\sin(3t)$$

So,

$$\mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{s^2+9}\right) = -\frac{1}{3}u(t-3\pi)\sin(3t)$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+9)^2}\right) = \frac{1}{6}t\sin(3t)$$

$$\Rightarrow \boxed{x(t) = \frac{t\sin(3t) - 2u(t-3\pi)\sin(3t)}{6}}$$



7.6.11

Apply Duhamel's principle to write an integral formula for the solution of each initial value problem.

$$x'' + 6x' + 8x = f(t) \qquad x(0) = x'(0) = 0.$$

$$\Rightarrow \quad s^2 X(s) + 6s X(s) + 8X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2 + 6s + 8} = W(s) F(s)$$

where
$$W(s) = \frac{1}{s^2 + 6s + 8} = \frac{1}{(s+3)^2 - 1}$$

$$\cancel{= \frac{\;}{e^{-3t}}}$$

$$\mathcal{L}^{-1}\{W(s)\} = e^{-3t}\sinh(t)$$

And so,

$$x(t) = \int_0^t e^{-3\tau}\sinh(\tau)\, f(t-\tau)\, d\tau$$

The convolution of ~~$\mathcal{L}^{-1}\{W(s)$~~ $\mathcal{L}^{-1}(W(s))$ and $f(t)$.

7.6.14. Verify that $u'(t-a) = \delta(t-a)$ by solving the problem

$$x' = \delta(t-a); \quad x(0) = 0$$

to obtain $x(t) = u(t-a)$.

Taking the Laplace transform of both sides:

$$s\,X(s) = e^{-as}$$

$$X(s) = \frac{e^{-as}}{s}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{e^{-as}\left(\tfrac{1}{s}\right)\}$$

$$= u(t-a)\,f(t-a)$$

$$f(t) = \mathcal{L}^{-1}\left(\tfrac{1}{s}\right) = 1.$$

So, $\boxed{x(t) = u(t-a)}$

7.6.15. This problem deals with a mass m on a spring (with constant k) that receives an impulse $p_0 = mv_0$ at time $t=0$.

Show that the initial value problems

$$mx'' + kx = 0; \qquad x(0) = 0, \quad x'(0) = v_0$$

and

$$mx'' + kx = p_0\,\delta(t); \quad x(0) = 0, \quad x'(0) = 0$$

have the same solution. Thus the effect of $p_0\,\delta(t)$ is, indeed, to impart to the particle an initial momentum p_0.

The first problem has the solution:

$$c_1 \cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$x(0) = c_1 = 0$. So, $c_1 = 0$.

$$x(t) = c_2 \sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$$x'(t) = c_2 \sqrt{\tfrac{k}{m}} \cos\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$$x'(0) = c_2 \sqrt{\tfrac{k}{m}} = v_0 \Rightarrow c_2 = v_0 \sqrt{\tfrac{m}{k}}.$$

So,

$$\boxed{x(t) = v_0 \sqrt{\tfrac{m}{k}} \sin\left(\sqrt{\tfrac{k}{m}}\,t\right)}$$

On the other hand

$$mx'' + kx = p_0\,\delta(t)\,; \quad x(0)=0,\ x'(0)=0.$$

taking the Laplace transform we get:

$$m s^2 X(s) + k X(s) = p_0$$

$$\Rightarrow \quad X(s) = \frac{p_0}{ms^2 + k} = \frac{p_0/m}{s^2 + k/m}$$

$$= \frac{p_0}{m}\sqrt{\frac{m}{k}}\left(\frac{\sqrt{k/m}}{s^2 + k/m}\right) \qquad p_0 = m v_0$$

$$= v_0 \sqrt{\frac{m}{k}}\left(\frac{\sqrt{k/m}}{s^2 + k/m}\right)$$

$$\mathcal{L}^{-1}(X(s)) = \boxed{v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)}$$

So, the ODEs have the same solution.