Quiz 2 Review Correction

Math 2280

Spring 2008

I screwed up the last example in the review session today. Sorry. I blindly picked one and goofed. So, very sorry about that, and here's a post with both a fully worked problem that is similar to what you might expect to see on the exam, along with the problem I did at the end of class, fully worked out and correct. Again, sorry.

Problem 5.4.5
Find general solutions of the system of ODEs described by the following matrix equation:

$$\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \mathbf{x}$$

To do this we want to first find the eigenvalues of the coefficient matrix. Setting up the characteristic equation:

$$\begin{vmatrix} 7 - \lambda & 1 \\ -4 & 3 - \lambda \end{vmatrix} = (7 - \lambda)(3 - \lambda) - (1)(-4) = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2$$

So, the matrix has a repeated eigenvalue $\lambda = 5$.

So, to get the corresponding solutions we first need to find an eigenvector for the eigenvalue $\lambda = 5$. To do this we solve the eigenvector equation:

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Or, more specifically,

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5a \\ 5b \end{pmatrix}$$

where the eigenvector $\mathbf{v}$ is:

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Now, solving this system for the vector $\mathbf{v}$ we get:

$$7a + b = 5a$$
$$-4a + 3b = 5b$$

which requires that $2a = -b$. So, a simple choice for our eigenvector would be:

$$\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now we note that we cannot find another linearly independent eigenvector. So, we need to construct a chain of eigenvectors. Namely, we need to find another vector $\mathbf{v}_2$ such that:

$$(\mathbf{A} - 5\mathbf{I})^2 \mathbf{v}_2 = 0$$

and

$$(\mathbf{A} - 5\mathbf{I}) \mathbf{v}_2 = \mathbf{v}_1$$

where given this construction it would have to be that $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$.

Now, we have:

$$(\mathbf{A} - 5\mathbf{I}) = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$$

and

$$(\mathbf{A} - 5\mathbf{I})^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, *any* vector $\mathbf{v}_2$ will work here so long as $(\mathbf{A} - 5\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$. So, for example, we could use:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with corresponding $\mathbf{v}_1$:

$$\mathbf{v}_1 = (\mathbf{A} - 5\mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

Now, our final solution will then be a linear combination of the two independent solutions:

$$\mathbf{x}_1 = \mathbf{v}_1 e^{5t}$$

and

$$\mathbf{x}_2 = (\mathbf{v}_1 t + \mathbf{v}_2) e^{5t}$$

You should know how to do this type of problem for the exam, but don't worry too much about larger chains, as they take too long.

However, just for the sake of completeness, here is the problem that we did in class, only this time done right.

Problem 5.4.14

Same problem, only this time with the system of ODEs given by:

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{pmatrix} \mathbf{x}$$

Now, if we solve for the eigenvalues of the matrix we get the characteristic equation:

$$\begin{vmatrix} -\lambda & 0 & 1 \\ -5 & -1-\Lambda & -5 \\ 4 & 1 & -2-\lambda \end{vmatrix} = -(\lambda + 1)^3$$

So, we've got a third-order eigenvalue $\lambda = -1$.

Now, if we solve the corresponding eigenvector equation we get:

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$$
\begin{pmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix}
$$

which, if we solve for a, b, c we get the relations: $a = -c$ and $4a + b - c = 0$. So, if we set $a = 1$ then we necessarily get $c = -1$ and $b = -5$. So, there's only one degree of freedom here, and therefore only one linearly independent eigenvector. So, we have a case of defective eigenvalues, and we need to construct a three step chain. That is, we need to find a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that satisfy the relations:

$$
\begin{aligned}
(\mathbf{A} - \lambda \mathbf{I})^3 \mathbf{v}_3 &= (\mathbf{A} + \mathbf{I})^3 \mathbf{v}_3 = 0 \\
(\mathbf{A} - \lambda \mathbf{I}) \mathbf{v}_3 &= (\mathbf{A} + \mathbf{I}) \mathbf{v}_3 = \mathbf{v}_2 \\
(\mathbf{A} - \lambda \mathbf{I}) \mathbf{v}_2 &= (\mathbf{A} + \mathbf{I}) \mathbf{v}_2 = \mathbf{v}_1
\end{aligned}
$$

Now, calculating these out we get:

$$
(\mathbf{A} + \mathbf{I}) = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -3 \end{pmatrix}
$$

$$
(\mathbf{A} + \mathbf{I})^2 = \begin{pmatrix} 5 & 1 & 0 \\ -25 & -5 & 0 \\ -5 & -1 & 0 \end{pmatrix}
$$

$$
(\mathbf{A} + \mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

So, noting that $(\mathbf{A} + \mathbf{I})^3 = 0$ we see that we can use *any* vector $\mathbf{v}_3$ so long as the corresponding $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-zero. So, if we start with, for example:

$$
\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

then we get:

$$\mathbf{v}_2 = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

and

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix}$$

So, now that we've build our chain, we get three linearly independent solutions from it:

$$\mathbf{x}_1 = \mathbf{v}_1 e^{-t}$$
$$\mathbf{x}_2 = (\mathbf{v}_1 t + \mathbf{v}_2) e^{-t}$$
$$\mathbf{x}_3 = (\frac{1}{2}\mathbf{v}_1 t^2 + \mathbf{v}_2 t + \mathbf{v}_3) e^{-t}$$

and so these are the three linearly independent solutions from which we construct our general solution. Any particular solution will be just a linear combination of these three solutions.

Again, sorry I messed this up in the review section. I hope this helps.