Math 2280 - Quiz 3

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Name: Solutions

50 Points Possible

Note - For credit you must show your work on all of these problems. A solution, even a correct solution, with no work or essentially no work will receive very little credit.

1. Calculate the following: (9 points)

a) Using the formal definition of the Laplace transform (i.e. calculate the integral) what is the Laplace transform of the function:

$$f(t) = t - 2e^{3t}$$

also state the domain of the Laplace transform. (3 points)

$$\int_0^\infty e^{-st}(t-2e^{3t})\,dt = \int_0^\infty t e^{-st}dt - 2\int_0^\infty e^{(3-s)t}dt$$

$$= -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\Big|_0^\infty - \frac{2e^{(3-s)t}}{(3-s)}\Big|_0^\infty$$

$$= \frac{1}{s^2} + \frac{2}{3-s} = \boxed{\frac{1}{s^2} - \frac{2}{s-3}}$$

Domain: $s > 3$

b) Calculate the convolution $f(t) * g(t)$ of the functions:

$$f(t) = t^2 \text{ and } f(t) = \cos t$$

(3 points)

$$(f * g)(t) = \int_0^t \tau^2 \cos(t-\tau)\, d\tau$$

$u = \tau^2 \qquad du = 2\tau\, d\tau$

$dv = \cos(t-\tau)\, d\tau \qquad v = -\sin(t-\tau)$

$$= -\tau^2 \sin(t-\tau)\Big|_0^t + 2\int_0^t \tau \sin(t-\tau)\, d\tau$$

$$= 2\int_0^t \tau \sin(t-\tau)\, d\tau$$

$u = \tau \qquad du = d\tau$

$dv = \sin(t-\tau) \qquad v = \cos(t-\tau)$

$$\Rightarrow \quad 2\tau\cos(t-\tau)\Big|_0^t - 2\int_0^t \cos(t-\tau)\, d\tau$$

$$= 2t + 2\sin(t-\tau)\Big|_0^t$$

$$= \boxed{2t - 2\sin t}$$

c) Again using the formal definition calculate the Laplace transform:

$$f(t) = t^2$$

again state the domain of the Laplace transform. (3 points)

$$\mathcal{L}(t^2) = \int_0^\infty t^2 e^{-st}\,dt$$

$u = t^2 \qquad du = 2t\,dt$

$dv = e^{-st}dt \qquad v = -\frac{e^{-st}}{s}$

$$\Rightarrow \int_0^\infty t^2 e^{-st}\,dt = -\frac{t^2 e^{-st}}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty t e^{-st}\,dt$$

$$= \frac{2}{s}\int_0^\infty t e^{-st}\,dt$$

$u = t \qquad du = dt$

$dv = e^{-st}dt \qquad v = -\frac{e^{-st}}{s}$

$$= -\frac{t e^{-st}}{s}\Big|_0^\infty + \frac{2}{s^2}\int_0^\infty e^{-st}\,dt$$

$$= 0 - \frac{2e^{-st}}{s^3}\Big|_0^\infty = \boxed{\frac{2}{s^3}}$$

2. Solve the initial value problem for the function $x(t)$:

$$x'' + 4x' + 5x = \delta(t-\pi) + \delta(t-2\pi);$$
$$x(0) = 0,\ x'(0) = 2.$$

(8 points)

$$\mathcal{L}(x'') = s^2 X(s) - s\,x(0) - x'(0)$$
$$= s^2 X(s) - 2$$
$$\mathcal{L}(x') = s\,X(s) - x(0) = s\,X(s)$$
$$\mathcal{L}(x) = X(s)$$
$$\mathcal{L}(\delta(t-\pi)) = e^{-\pi s} \qquad \mathcal{L}(\delta(t-2\pi)) = e^{-2\pi s}.$$

So, the Laplace transform of the equation is:

$$s^2X(s) - 2 + 4sX(s) + 5X(s) = e^{-\pi s} + e^{-2\pi s}$$
$$\Rightarrow (s^2+4s+5)X(s) = 2 + e^{-\pi s} + e^{-2\pi s}$$
$$\Rightarrow X(s) = \frac{2}{(s+2)^2+1} + \frac{e^{-\pi s}}{(s+2)^2+1} + \frac{e^{-2\pi s}}{(s+2)^2+1}$$

Now, $\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+1}\right) = e^{-2t}\sin(t)$

and $\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a)f(t-a)$.

So,

$$\boxed{\begin{aligned} x(t) = {} & 2e^{-2t}\sin(t) + u(t-\pi)\,e^{-2(t-\pi)}\sin(t-\pi) \\ & + u(t-2\pi)\,e^{-2(t-2\pi)}\sin(t-2\pi) \end{aligned}}$$

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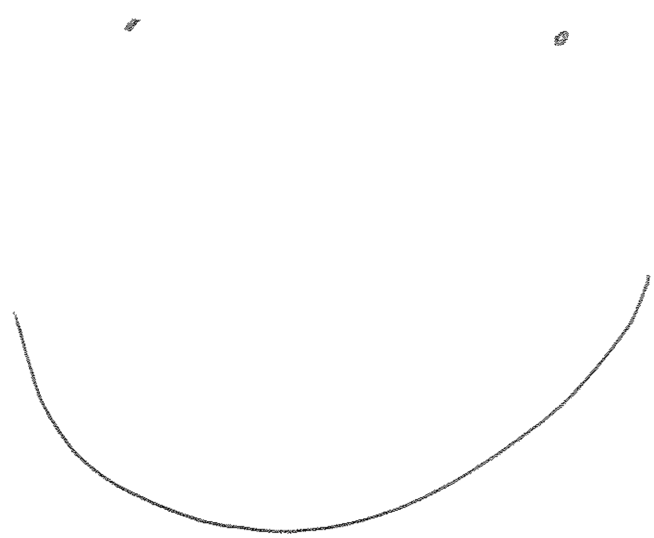

3. Determine if $x = 0$ is an ordinary, regular singular, or irregular singular point in each of the following differential equations: (9 points)

a) (3 points)

$$3x^3y'' + 2x^2y' + (1 - x^2)y = 0$$

Rewriting

$$y'' + \frac{2}{3x}y' + \frac{(1-x^2)}{3x^3}y = 0$$

$\lim_{x \to 0} \left(\frac{2}{3x}\right)$ is undefined, so the point is singular.

Now,

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{x^2(1-x^2)}{3x^3} = \lim_{x \to 0} \frac{(1-x^2)}{3x}$$

is still undefined.

So, $x=0$ is an irregular singular point.

b) (3 points)

$$x^2(1-x^2)y'' + 2xy' - 2y = 0$$

Rewriting

$$y'' + \frac{2}{x(1-x^2)}y' - \frac{2}{x^2(1-x^2)}y = 0$$

$\lim_{x\to 0} \frac{2}{x(1-x^2)}$ is undefined, so $x=0$ is a singular point.

$$p(x) = x\left(\frac{2}{x(1-x^2)}\right) = \frac{2}{1-x^2}$$

$$\lim_{x\to 0}\left(\frac{2}{1-x^2}\right) = 2.$$

$$q(x) = x^2\left(-\frac{2}{x^2(1-x^2)}\right) = -\frac{2}{1-x^2}$$

$$\lim_{x\to 0} q(x) = -2.$$

Both limits are defined, so $x=0$ is a <u>regular singular point.</u>

c) (3 points)

$$xy'' + x^2y' + (e^x - 1)y = 0$$

Rewriting:

$$y'' + xy' + \left(\frac{e^x-1}{x}\right)y = 0$$

$$\lim_{x\to 0} x = 0.$$

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = \frac{0}{0} \quad \text{apply L'Hospital's rule}$$

$$= \lim_{x\to 0}\left(\frac{e^x}{1}\right) = 1.$$

So, both limits at $x=0$ for the coefficient functions are defined. So, $x=0$ is an <u>ordinary point</u>.

4. Solve the following second-order ODE using power series or Frobenius series methods:

$$y'' + x^2y' + 2xy = 0$$

(8 points)

$\lim_{x \to 0} x^2 = 0$ $\lim_{x \to 0} 2x = 0.$

So, $x = 0$ is an ordinary point.

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

Plugging these into our ODE we get:

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^{n+1} + 2 \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

The first non-zero, or at least not automatically 0, power of x is x^0. For this power $n=2$ in the first series, while the other series don't come in. So,

$$2(2-1) c_2 x^0 = 0x^0. \Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0$$

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Now, c_0 and c_1 are "arbitrary" and for higher order powers we get the recurrence relation:

$$(n+3)(n+2)c_{n+3} + (n+2)c_n = 0$$

$$\Rightarrow \quad c_{n+3} = \frac{-(n+2)c_n}{(n+3)}$$

as $n \geq 0$

$$\Rightarrow \quad c_{n+3} = \frac{-c_n}{n+3}.$$

So,

$$c_0 = c_0 \qquad c_1 = c_1 \qquad c_2 = 0$$

$$c_3 = -\frac{c_0}{3} \qquad c_4 = -\frac{c_1}{4} \qquad c_5 = 0$$

$$c_6 = -\frac{c_3}{6} = \frac{c_0}{6\cdot 3} \qquad c_7 = -\frac{c_4}{7} = \frac{c_1}{7\cdot 4}$$

In general:

$$c_{3n+2} = 0.$$

$$c_9 = -\frac{c_6}{9} = -\frac{c_0}{9\cdot 6\cdot 3} \qquad c_{10} = -\frac{c_7}{10} = \frac{-c_1}{10\cdot 7\cdot 4}$$

In general

$$c_{3n} = \frac{c_0(-1)^n}{3^n\, n!}$$

In general:

$$c_{3n+1} = \frac{c_1(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)}$$

So, our solution will be:

$$\boxed{y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{3^n\, n!} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{1\cdot 4\cdots(3n+1)}}$$

5. Solve the following second-order ODE using power series or Frobenius series methods:

$$2xy'' - y' - y = 0$$

(8 points)

<u>Rewriting</u>

$\lim\limits_{x \to 0} \left(-\frac{1}{2x}\right)$ is undefined, so $x=0$ is a singular point.

$$\lim_{x \to 0} p(x) = \lim_{x \to 0} x\left(-\frac{1}{2x}\right) = -\frac{1}{2} = p_0.$$

$$\lim_{x \to 0} q(x) = \lim_{x \to 0} x^2\left(-\frac{1}{2x}\right) = 0 = q_0.$$

Both limits are defined, so $x=0$ is a regular singular point.

So, we have the indicial equation:

$$r(r-1) - \frac{1}{2}r = 0 \Rightarrow r = \{\tfrac{3}{2}, 0\}.$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

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Plugging this into the ODE we get:

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

So, we get c_0 is "arbitrary" and for higher coefficients we get the recurrence relation:

$$[2(n+r+1)(n+r) - (n+r+1)]c_{n+1} - c_n = 0.$$

$$\Rightarrow \quad c_{n+1} = \frac{c_n}{(2n+2r-1)(n+r+1)}$$

For $r = 3/2$

$$c_{n+1} = \frac{c_n}{(2n+2)(n+5/2)} = \frac{c_n}{(n+1)(2n+5)}$$

So,

$$c_0 = c_0$$

$$c_1 = \frac{c_0}{5}$$

$$c_2 = \frac{c_1}{2\cdot 7} = \frac{c_0}{(2\cdot 1)(7\cdot 5)}$$

$$c_3 = \frac{c_2}{3\cdot 9} = \frac{c_0}{(3\cdot 2\cdot 1)(9\cdot 7\cdot 5)}$$

and in general for $n \geq 1$

$$c_n = \frac{3c_0}{n!(2n+3)!!}$$

For $r = 0$

$$c_{n+1} = \frac{c_n}{(2n-1)(n+1)}$$

So,

$$c_0 = c_0$$

$$c_1 = -c_0$$

$$c_2 = \frac{c_1}{1\cdot 2} = -\frac{c_0}{1\cdot 2}$$

$$c_3 = \frac{c_2}{3\cdot 3} = -\frac{c_0}{(1\cdot 3)(3\cdot 2\cdot 1)}$$

$$c_4 = \frac{c_3}{5\cdot 4} = \frac{-c_0}{(1\cdot 3\cdot 5)(4\cdot 3\cdot 2\cdot 1)}$$

and in general for $n \geq 2$

$$c_n = -\frac{c_0}{(2n-3)!!\,n!}$$

So, our final solution is:

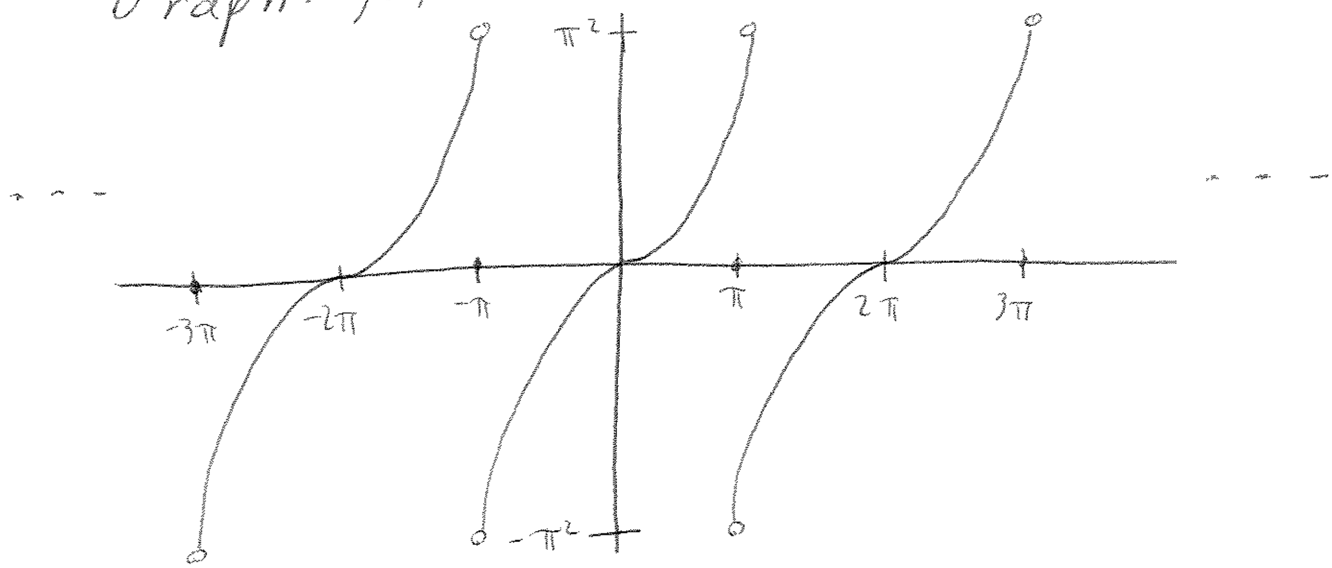
$$\boxed{y(x) = c_1 x^{3/2}\left(1 + 3\sum_{n=1}^{\infty}\frac{x^n}{n!(2n+3)!!}\right) + c_2\left(1 - x - \sum_{n=2}^{\infty}\frac{x^n}{n!(2n-3)!!}\right)}$$

6. Find the odd extension of the function defined below, and graph the odd extension. Then, calculate the corresponding Fourier series (sine series) representation of the odd extension.

$$f(t) = t^2, 0 < t < \pi.$$

(8 points)

Graph: $g(t)$



The Fourier sine series will have coefficients:

$$b_n = \frac{2}{\pi}\int_0^{\pi} t^2 \sin\left(\frac{n\pi t}{\pi}\right) dt$$

$$= \frac{2}{\pi}\int_0^{\pi} t^2 \sin(nt)\, dt$$

$u = t^2 \qquad dv = \sin(nt)dt$

$du = 2t\,dt \qquad v = -\frac{\cos(nt)}{n}$

Continued

So,

$$\frac{2}{\pi}\int_0^{\pi} t^2 \sin(nt)\,dt = \frac{-2t^2\cos(nt)}{n\pi}\Big|_0^{\pi} + \frac{4}{n\pi}\int_0^{\pi} t\cos(nt)\,dt$$

$$= \frac{-2\pi}{n}\cos(n\pi) + \frac{4}{n\pi}\int_0^{\pi} t\cos(nt)\,dt$$

$u = t \qquad dv = \cos(nt)\,dt$

$du = dt \qquad v = \frac{\sin(nt)}{n}$

$$= -\frac{2\pi}{n}\cos(n\pi) + \frac{4}{n^2\pi}\,t\sin(nt)\Big|_0^{\pi} - \frac{4}{n^2\pi}\int_0^{\pi}\sin(nt)\,dt$$

$$= -\frac{2\pi}{n}\cos(n\pi) + 0 + \frac{4}{n^3\pi}\cos(nt)\Big|_0^{\pi}$$

$$= \frac{4}{n^3\pi}\left(\cos(n\pi) - 1\right) - \frac{2\pi}{n}\cos(n\pi)$$

Now,

$$\cos(n\pi) = (-1)^n$$

So, our Fourier series breaks up as:

$$\boxed{\begin{aligned} FS(g(t)) = {} & 2\pi\left(\sin(t) - \tfrac{1}{2}\sin(2t) + \tfrac{1}{3}\sin(3t) - \tfrac{1}{4}\sin(4t) + \cdots\right) \\ & - \frac{8}{\pi}\left(\sin(t) + \frac{1}{3^3}\sin(3t) + \frac{1}{5^3}\sin(5t) + \frac{1}{7^3}\sin(7t) + \cdots\right) \end{aligned}}$$

7. (Extra Credit) Derive the following equivalence:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

where the product on the right is over prime numbers p. In your derivation you can use the relation:

$$\sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \left(1 - \frac{1}{p^s}\right)^{-1}.$$

(5 points)

We write

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \frac{1}{p_1^{k_1 s} p_2^{k_2 s} \cdots}$$

noting ~~ea~~ the bijection between products of primes and the integers. Well

$$\zeta(s) = \sum_{k_1=0}^{\infty} \frac{1}{p_1^{k_1 s}} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \frac{1}{p_2^{k_2 s} p_3^{k_3 s} \cdots}$$

$$= \left(1 - \frac{1}{p_1^s}\right)^{-1} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{p_2^{k_2 s} p_3^{k_3 s} \cdots}$$

Repeat for each prime to get

$$= \prod_p \left(1 - \frac{1}{p_i^s}\right)^{-1}$$

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