Quiz 3 Review Notes

Math 2280

Spring 2008

Disclaimer - These notes are designed to be just that, notes, for a review of what we've studied since the second exam. They are certainly not exhaustive, but will provide an outline of the material that you will need to know for the third exam. They are intended as a study aid, but they should definitely not be the only thing you study. I also cannot guarantee they are free from typos, either in the exposition or in the equations. Please refer to the appropriate equations in the textbook while you're studying to make sure the equations in these notes are free of typos, and if you find a typo please contact me and let me know. If there are any sections upon which you're unclear or think you need more information, please read the appropriate section of the book or you lectures notes. If after doing this it's still unclear please ask me, either at my office, via email, or at the review session. If you think you need further practice in any area I'd recommend working the recommended problems in the textbook from the sections that cover the material on which you're having trouble.

1 Laplace Transforms

1.1 The Laplace Transform

The Laplace transform is a mapping from functions to functions. That is, you take a function as your input, you apply the Laplace transform, and a different function comes out as your output.

The Laplace transform is defined as:

Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined as follows:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

for all values of s for which the improper integral converges.

1.2 Properties of Laplace Transforms

Now, the Laplace transform has some very useful properties:

- **Existance and Uniqueness**

  If the function f is piecewise continuous for $t \geq 0$ and is of exponential order as $t \to +\infty$, then its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists. More precisely, if f is piecewise continuous and satisfies the condition in (23), then $F(s)$ exists for all $s > c$.

  If the functions $f(t)$ and $g(t)$ satisfy the existance conditions mentioned in the last paragraph then the Laplace transforms $F(s)$ and $G(s)$ exist. If $F(s) = G(s)$ for all $s > c$, then $f(t) = g(t)$ wherever on $[0, \infty)$ both f and g are continuous.

- **Linearity**

  If a and b are constants, then

  $$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

  for all s such that the Laplace transforms of the functions f and g both exist.

- **Transforms of Derivatives and Integrals**

  Suppose that the function $f(t)$ is continuous and piecewise smooth for $t \geq 0$ and is of exponential order as $t \to +\infty$, so that there exist nonnegative constants M, c, and T such that

  $$|F(t)| \leq Me^{ct} \text{ for } t \geq T.$$

  Then $\mathcal{L}\{f'(t)\}$ exists for $s > c$, and

  $$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0).$$

and in general if the functions $f, f', f'', \ldots, f^{(n-1)}$ are continuous and piecewise smooth for $t \geq 0$, and that each of these functions satisfies the conditions specified above with the same values of M and c, then $\mathcal{L}\{f^{(n)}(t)\}$ exists when $s > c$, and

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) =$$
$$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

In a simiar fashion we have that if $f(t)$ is a piecewise continuous function for $t \geq 0$ and satisfies the condition of exponential order $|f(t)| \leq Me^{ct}$ for $t \geq T$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

- **Convolution**

  The convolution $f * g$ of the piecewise continuous functions f and g is defined for $t \geq 0$ as follows:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

  Now, suppose that $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$ and that $|f(t)|$ and $|g(t)|$ are bounded by Me^{ct} as $t \to +\infty$. Then the Laplace transform of the convolution $f(t) * g(t)$ exists for $s > c$; moreover,

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

  and

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t).$$

- **Differentiation and Integration of Transforms**

  If $f(t)$ is piecewise continuous for $t \geq 0$ and $|f(t)| \leq Me^{ct}$ as $t \to +\infty$, then

$$\mathcal{L}\{-tf(t)\} = F'(s)$$

for $s > c$.

Repeated application of this relation gives:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

for $n = 1, 2, 3, \ldots$.

Conversely, suppose that $f(t)$ is piecewise continuous for $t \geq 0$, that $f(t)$ satisfies:

$$\lim_{t \to 0^+} \frac{f(t)}{t}$$

exists and is finite

and that $|f(t)| \leq Me^{ct}$ as $t \to \infty$. Then,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma) d\sigma$$

for $s > c$.

- **Translation on the t-Axis**

  If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

  $$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

  for $s > c + a$.

  Here, the function $u(t)$ is defined as:

  $$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

- **Transforms of Periodic Functions**

  Let $f(t)$ be periodic with period p and piecewise continuous for $t \geq 0$. Then the transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > 0$ and is given by

$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)dt.$$

- **Laplace Transform of the Delta Function**

  The delta function is the function $\delta(t)$ that is *defined* by its Laplace transform:

  $$\mathcal{L}\{\delta(t)\} = 1$$

  and by extension:

  $$\mathcal{L}\{\delta(t-a)\} = e^{-at}$$

  Note that the delta function is not actually a "function" per se, it's actually a "generalized function" of a "distribution" as it is only defined in the context of its Laplace transform.

1.3 Laplace Transforms of Common Functions

The Laplace transforms of many common functions are listed on the back of the front cover of the textbook, to which you will have access on the exam. By far the most useful and most common relations are:

$$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}} \text{ for } (s > a).$$

$$\mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2 + k^2} \text{ for } (s > a).$$

$$\mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2 + k^2} \text{ for } (s > a).$$

These relations are so useful because they show up when taking the inverse Laplace transform of a rational function after a partial fraction decomposition. Note that you should *definite* understand and know how to do partial fraction decompositions for the exam.

1.4 Examples

- Find the Laplace transform of e^{3t+1}.

  Solution

$$\mathcal{L}\{e^{3t+1}\} = \int_0^\infty e^{-st}e^{3t+1}dt = \int_0^\infty e^{(3-s)t+1}dt = \frac{e^{(3-s)t+1}}{3-s}|_0^\infty$$
$$= \frac{e}{s-3} \text{ for } s > 3.$$

- Use Laplace transforms to solve the initial value problem:

  $x^{(4)} + 8x'' + 16x = 0$; $x(0) = x'(0) = x''(0) = 0$, $x^{(3)}(0) = 1$.

  If we take the Laplace transform of the terms on the left we get:

$$\mathcal{L}\{x^{(4)}\} = s^4X(s) - s^3x(0) - s^2x'(0) - sx''(0) - x^{(3)}(0) = s^4X(s) - 1$$
$$\mathcal{L}\{8x''\} = 8s^2X(s) - 8sx(0) - 8x'(0) = 8s^2X(s)$$
$$\mathcal{L}\{16x\} = 16X(s)$$

  which means the Laplace transform of the equation is:

$$(s^4 + 8s^2 + 16)X(s) - 1 = 0$$
$$\rightarrow X(s) = \frac{1}{s^4 + 8s^2 + 16} = \frac{1}{(s^2+4)^2}.$$

  Now, using equation 17 from section 7.3 of the textbook (which will be available to you on the exam, and the equation also appears on the back of the front cover) we see that the inverse Laplace transform is:

$$x(t) = \frac{1}{16}(\sin 2t - 2t\cos 2t)$$

  which is the solution to our ODE.

- Use Laplace transforms to solve the initial value problem:

$$x'' + 4x' + 4x = 1 + \delta(t-2); x(0) = x'(0) = 0$$

Solution

Taking the Laplace transform of both sides we get:

$$(s^2 + 4s + 4)X(s) = \frac{1}{s} + e^{-2s}$$
$$\to X(s) = \frac{1}{s(s+2)^2} + \frac{e^{-2s}}{(s+2)^2}$$

Now, if we take a partial fraction decomposition of the right hand side we get:

$$X(s) = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2} + \frac{e^{-2s}}{(s+2)^2}.$$

Now, we can figure out the inverse Laplace transform of this using the three major inverse Laplace transforms discussed earlier along with the translation of the t-axis formula:

$$x(t) = \frac{1}{4} - \frac{e^{-2t}}{4} - \frac{te^{-2t}}{2} + u(t-2)(t-2)e^{-2(t-2)}$$

which is the solution to our ODE.

2 Power Series Methods

A power series is a representation of a function as a potentially infinite series of powers of the variable x:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

The series is defined for all values x where the series converges. The coefficients c_n completely determine the function $f(x)$.

The power series method basically boils down to assuming that the solution to the differential equation can be represented as a power series, and then figuring out what the coefficients must be in order for the function they define to satisfy the requisite ODE.

2.1 Important Theorems

There are really three major theorems that form the basis for most of the power series methods.

- **Termwise Differentiation**

  If the power series representation

  $$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

  of the function f converges on the open interval I, then f is differentiable on I, and

  $$f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

  at each point of I.

- **Identity Principle**

  If

  $$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

  for every point x is some open interval I, then $a_n = b_n$ for all $n \geq 0$.

- **Radius of Convergence**

  Given the power series $\sum c_n x^n$, suppose that the limit

$$\rho = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

exists (ρ is finite) or is infinite. Then

1. If $\rho = 0$, then the series diverges for all $x \neq 0$.
2. If 0 ¡ ρ ¡ ∞, then $\sum c_n x^n$ converges if $|x| < \rho$ and diverges if $|x| > \rho$.
3. If $\rho = \infty$, then the series converges for all x.

2.2 Ordinary Points, Regular Singular Points

Most of the examples from the text, and hence all of the questions on the exam, that require the use of power series methods involve homogenous second-order linear differential equations:

$$A(x)y'' + B(x)y' + C(x)y = 0$$

Now, we assume that $A(x) \neq 0$, or else it would be a first-order differential equation. As this is the case we can rewrite the above as:

$$y'' + P(x)y' + Q(x)y = 0$$

Now, $P(x)$ and $Q(x)$ may fail to be analytic at points where $A(x) = 0$.

2.2.1 Ordinary Points

If a is an ordinary point of the equation

$$A(x)y'' + B(x)y' + C(x)y = 0$$

this means that the functions $P(x) = B(x)/A(x)$ and $Q(x) = C(x)/A(x)$ are analytics at $x = a$. If this is the case them the differential equation has two linearly independent solutions, each of the form:

$$y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

The radius of convergence of any such series solutions is at least as large as the distance from a to the nearest (real or complex) singular point of the differential equation. The coefficients c_n can be determined by substitution of the series in to the differential equation and the subsequent derivation of the recurrence relations.

2.2.2 Regular Singular Points

For these problems we'll assume that we're interested in solutions around the point $x = 0$. We can shift any second order ODE so that a point of interest becomes the pont $x = 0$, so there is really no loss of generality in focusing only on this case.

If the functions $P(x)$ or $Q(x)$ defined above are not analytic at $x = 0$ then $x = 0$ is a singular point. We call $x = 0$ a *regular* singular point if $p(x) = xP(x)$ and $q(x) = x^2Q(x)$ are both analytic at $x = 0$. Otherwise, the point $x = 0$ is called an irregular singular point.

If $x = 0$ is a regular singular point then we can rewrite the differential equation as:

$$x^2y'' + xp(x)y' + q(x)y = 0$$

and solve this equivalent differential equation.

If $x = 0$ is a regular singular point then if we let $\rho > 0$ denote the minimum of the radii of convergence of the power series for $p(x)$ and $q(x)$ and r_1, r_2 ($r_1 > r_2$) be the roots of the indicial equation:

$$r(r-1) + p_0r + q_0 = 0 \text{ (where } p_0 = p(0) \text{ and } q_0 = q(0))$$

then:

1. For $x > 0$, there exists a solution of the ODE of the form:

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_nx^n$$

corresponding to the larger root r_1.

2. If $r_1 - r_2$ is neither zero nor a positive integer, then there exists a second linearly independent solution for $x > 0$ of the form

$$y_2(x) = x^{r_2}\sum_{n=0}^{\infty} b_nx^n$$

corresponding to the smaller root r_2.

The radii of convergence of the power series for both solutions are each at least ρ. The coefficients can be determined by substitution the series into the differential equation.

This is called the method of Frobenius, and the solutions are called Frobenius series solutions.

Now, if $r_1 - r_2$ is an integer it may be possible that we cannot get a second Frobenius series solution using the methods described above. We always get one solutions for the larger root r_1, call it $y_1(x)$. Using this first solution we find that the second solution is given by the formula:

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1(x)^2} dx.$$

Now, the particulars of this second are usually very messy, and I don't think I'll have any questions on the exam that involve using this nasty equation. Same goes for Bessel functions. So, in other words, don't worry too much about the material in sections 8.4 or 8.5. The only exception is that you should know the definition of the Gamma function and how to use it.

2.3 The Gamma Function

The gamma funciton $\Gamma(x)$ is defined for $x > 0$ by:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Now, we can use integration by parts to prove that:

$$\Gamma(x+1) = x\Gamma(x)$$

and in fact:

$$\Gamma(n+1) = n!.$$

I'd recommend knowing how to prove the above relations for the half of the final exam on which you're not allowed to use your book. Just a suggestion.

2.4 Examples

Here are two examples of how to use power series methods to solve ODEs, one that involves the standard method, and the other that involves the method of Frobenius. Many more examples can be found in the textbook.

- Solve the ODE $y'' + xy' + y = 0$

  Solution

  We assume that the solution is of the form:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

  the derivatives of this function will be:

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

  If we plug these into our differential equation we get:

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} (n+1) c_n x^n = 0$$

  which we can rewrite as:

$$\sum_{n=0} [(n+2)(n+1)c_{n+2} + (n+1)]x^n = 0$$

  We can see from this equation that c_0 and c_1 are "arbitrary" in that they're determined by whatever the initial values of the ODE are. The higher order terms are given by the recurrence relation:

$$c_{n+2} = \frac{-c_n}{n+2}$$

Now, as far as the exam is concerned, if you understand enough to get this far you'll get most of the points. However, we can derive a closed form solution for the odd and even coefficients, and therefore write the solution to the ODE out explicitly.

For the even terms we get:

$$\begin{aligned}
c_0 &= c_0 \\
c_2 &= \frac{-c_0}{2} \\
c_4 &= \frac{-c_2}{4} = \frac{c_0}{4(2)} \\
c_6 &= \frac{-c_4}{6} = \frac{c_0}{6(4)(2)}
\end{aligned}$$

We can see where this is going, and can deduce by induction (formal proof omitted, but it's easy) that:

$$c_{2n} = \frac{(-1)^n c_0}{2^n n!}$$

and in an essentially identical fashion we can deduce that:

$$c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!!}.$$

So, our final solution to the differential equation is:

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n c_1}{(2n+1)!!} x^{2n+1}$$

- Solve the ordinary differential equation:

$$3xy'' + 2y' + 2y = 0$$

Solution

We first note that if we rewrite this ODE as:

$$y'' + \frac{2}{3x}y + \frac{2}{3x}y = 0$$

that the point $x = 0$ is not an ordinary point. It is, however, a regular point, with $p(x) = \frac{2}{3}$ and $q(x) = \frac{2x}{3}$. So, $p_0 = \frac{2}{3}$ and $q_0 = 0$, which gives the indicial equation:

$$r(r-1) + \frac{2}{3}r + 0 = 0$$

which has roots: $r_1 = \frac{1}{3}$ and $r_2 = 0$.

The corresponding solutions are:

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

if we plug this into the ODE we get:

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + n + r] c_n x^{n+r-1} + c_n x^{n+r}$$

Now, the coefficient c_0 is arbitrary by design (this is where the indicial equation comes from) but the rest of the coefficients are given by the recurrence relation:

$$(n+r+1)^2 c_{n+1} + c_n = 0$$

or

$$c_{n+1} = \frac{-c_n}{(n+r+1)^2}.$$

Now, using this recurrence relation we can examine the first few terms and then derive a closed form solution for our two roots $r_1 = \frac{1}{3}$ and $r_2 = 0$. These solutions are:

$$y_1(x) = x^{\frac{1}{3}} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n! 4 * 7 * 10 \ldots (3n+1)}$$

and

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n! 2 * 5 \ldots (3n-1)}$$

3 Fourier Series

A Fourier series is a way of representing a periodic function as an infinite sum of sine and cosine functions.

The Fourier series for a periodic function $f(t)$ with period $2L$ is given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

where the Fourier coefficients are given by:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} dt$$

and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} dt.$$

If the function $f(t)$ is piecewise smooth then its Fourier series converges to the value $\frac{1}{2}[f(t^+) + f(t^-)]$, which is the value $f(t)$ at all points t where $f(t)$ is continuous.

3.1 Even and Odd Extensions

Frequently we only know, or at least are only interested in, the value of a function on a given interval, say (without loss of generality, as we can always just translate our axis) from 0 to L. Now, a function defined on this domain cannot be periodic, as periodic functions must be defined for all values t. However, we can create a new function $g(t)$ that agrees with $f(t)$ on the domain of $f(t)$ where $g(t)$ is periodic.

There are two standard ways of doing this. One is to take the even extension of $f(t)$, the other is to take the odd extension.

The even extension is defined as:

$$g(t) = \begin{cases} f(t) & \text{for } 0 < t < L \\ f(-t) & \text{for } -L < t < 0 \end{cases}$$

from $-L$ to L, and it is then required to be $2L$ periodic. This creates an even function defined on the whole real line that agrees with $f(t)$ on the domain of $f(t)$.

Similarly the odd extension is:

$$g(t) = \begin{cases} f(t) & \text{for } 0 < t < L \\ -f(-t) & \text{for } -L < t < 0 \end{cases}$$

from $-L$ to L, and it is then required again to be $2L$ periodic. This creates an odd function defined on the whole real line that agrees with $f(t)$ on the domain of $f(t)$.

Now, the Fourier series of the even extension will involve only cosine terms, while the Fourier series of the odd extension will involve only sine terms. These Fourier series are therefore sometimes referred to as the cosine and sine series of the function $f(t)$.

3.2 Example

I'll just do one example from the textbook. Here the value of a function along one of its periods is given, and we're asked to find the Fourier series for the function.

$$f(t) = t \text{ for } -2 < t < 2.$$

Solution

As t is an odd function the even coefficients in the Fourier series will all integrate to 0. So, the only non-zero terms will be the sine terms with coefficients given by:

$$b_n = \frac{1}{2}\int_{-2}^{2} t \sin\frac{n\pi t}{2} dt.$$

Now, first we note that the integrand is a product of two odd functions, so the integrand is an even function, and therefore we can rewrite it as:

$$b_n = \frac{1}{2}\int_{-2}^{2} t \sin\frac{n\pi t}{2} dt = \int_{0}^{2} t \sin\frac{n\pi t}{2} dt.$$

If we integrate this by parts we get:

$$b_n = \int_{0}^{2} t \sin\frac{n\pi t}{2} dt = \frac{4}{n^2\pi^2}\sin\frac{n\pi t}{2}|_0^2 - \frac{2t}{n\pi}\cos\frac{n\pi t}{2}|_0^2$$

which, when we plug in our limits and use the fact that $\sin n\pi = 0$ for all integers n we get:

$$\frac{4}{n^2\pi^2}\sin\frac{n\pi t}{2}|_0^2 - \frac{2t}{n\pi}\cos\frac{n\pi t}{2}|_0^2 = \frac{4(-1)^{n+1}}{n\pi}.$$

So, our Fourier series will be:

$$f(t) = \frac{4}{\pi}\left(\sin\frac{\pi t}{2} - \frac{1}{2}\sin \pi t + \frac{1}{3}\sin\frac{3\pi t}{2} - \dots\right).$$

Good Luck on the Exam!