

# Math 2280 - Quiz 2

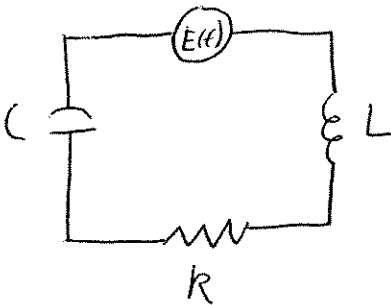
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Name: Solutions

**50 Points Possible**

1. The circuit below is being driven by a power source that provides an EMF given by  $E(t) = \cos 3t$ . The resistor's resistance is  $6\Omega$ , the inductor's inductance is  $1H$ , and the capacitor's capacitance is  $.04F$ . Find the form of the function  $I(t)$  (the general solution to the ODE) that describes the flow of current through the circuit. (10 points)



The ODE for the system is:

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = E'(t)$$

$$\Rightarrow \frac{d^2 I}{dt^2} + 6 \frac{dI}{dt} + 25 I = -3 \sin(3t)$$

The homogenous part has characteristic equation:  $r^2 + 6r + 25$

So,

$$r = \frac{-6 \pm \sqrt{6^2 - 4(1)(25)}}{2} = -3 \pm \frac{\sqrt{-64}}{2} = -3 \pm 4i.$$

So,

$$I_h = e^{-3t} (c_1 \cos(4t) + c_2 \sin(4t))$$

As for the particular solution we use undetermined coefficients

$$I_p = A \cos(3t) + B \sin(3t) \quad I_p' = -3A \sin(3t) + 3B \cos(3t)$$

$$I_p'' = -9A \cos(3t) - 9B \sin(3t)$$

So, we get the relations:

$$(-9A \cos(3t) - 9B \sin(3t)) + 6(-3A \sin(3t) + 3B \cos(3t)) + 25(A \cos(3t) + B \sin(3t))$$

$$= -3 \sin(3t). \quad \text{So,}$$

$$\begin{aligned} 16A + 18B &= 0 \\ -18A + 16B &= -3 \end{aligned} \Rightarrow A = \frac{27}{290} \quad B = -\frac{12}{145}$$

and

$$I(t) = I_h + I_p = e^{-3t} (c_1 \cos(4t) + c_2 \sin(4t)) + \frac{27}{290} \cos(3t) - \frac{12}{145} \sin(3t)$$

2. Convert the following system of equations into an equivalent system of first-order equations: (5 points)

$$x^{(3)} = x'' - 2x' + 5y' + 2x + 1$$

$$y'' = x' + 5x - 14y'$$

Create Variables

$$\begin{aligned} x' &= x_1 & y' &= y_1 \\ x_1' &= x_2 = x'' & y_1' &= y'' \\ x_2' &= x^{(3)} \end{aligned}$$

So, we get the first-order ODEs:

$$x_2' = x_2 - 2x_1 + 5y_1 + 2x + 1$$

$$x_1' = x_2$$

$$x' = x_1$$

$$y_1' = x_1 + 5x - 14y_1$$

$$y' = y_1$$

3. Calculate the Wronskian of the given vector-valued functions, and use it to determine if these functions are linearly independent.  
(5 points)

$$\mathbf{x}_1(t) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^t$$

$$\mathbf{x}_2(t) = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{3t}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{5t}$$

$$W(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \begin{vmatrix} 2e^t & -2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & e^{3t} & e^{5t} \end{vmatrix}$$

$$= 2e^t(2e^{8t}) + 2e^{3t}(4e^{6t}) + 2e^{5t}(2e^{4t})$$

$$= \boxed{16e^{9t}}$$

Now,  $16e^{9t} \neq 0$  at any time  $t$ , so the functions are linearly independent.

4. Find a general solution to the system of ODEs: (10 points)

$$x'_1 = 2x_1 - x_2 - x_3$$

$$x'_2 = -4x_1 - 3x_2 - x_3$$

$$x'_3 = 4x_1 + 4x_2 + 2x_3$$

$$\vec{x}' = \begin{pmatrix} 2 & -1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix} \vec{x} \Rightarrow \begin{vmatrix} 2-\lambda & -1 & -1 \\ -4 & -3-\lambda & -1 \\ 4 & 4 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)((-3-\lambda)(2-\lambda) - (-4)) + (-4(2-\lambda) - (-4)) - (-16 - (-3-\lambda)4)$$

$$= (2-\lambda)(\lambda^2 + \lambda - 2) + (4\lambda - 4) - (4\lambda - 4)$$

$$= (2-\lambda)(\lambda+2)(\lambda-1) \quad \lambda = 1, 2, -2 \text{ are eigenvalues}$$

The associated ~~eigenvalue~~ eigenvector is: for  $\lambda = 1$

$$\begin{pmatrix} 2 & -1 & -1 \\ -4 & -3 & -1 \\ -4 & 4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{cases} a - b - c = 0 \\ -4a - 4b - c = 0 \\ 4a + 4b + c = 0 \end{cases}$$

an associated eigenvector would be:  $\begin{pmatrix} 3 \\ -5 \\ 8 \end{pmatrix}$

The eigenvector associated with  $\lambda = 2$

$$\begin{pmatrix} 2 & -1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \Rightarrow \begin{array}{l} -b - c = 0 \\ -4a - 3b - c = 0 \\ 4a + 4b = 0 \end{array} \quad \begin{array}{l} a = 1 \\ b = -1 \\ c = 1 \end{array} \text{ works}$$

So, an associated eigenvector would be:  $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

The eigenvector associated with  $\lambda = -2$

$$\begin{pmatrix} 2 & -1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a \\ -2b \\ -2c \end{pmatrix} \Rightarrow \begin{array}{l} 4a - b - c = 0 \\ -4a - b - c = 0 \\ 4a + 4b + 4c = 0 \end{array} \quad \begin{array}{l} a = 0 \\ b = 1 \\ c = -1 \end{array} \text{ works}$$

So, an associated eigenvector would be:  $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

A general solution would then be:

$$\vec{X}(t) = c_1 \begin{pmatrix} 3 \\ -5 \\ 8 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t}$$

5. Find a fundamental matrix for the system of ODEs described below:  
(5 points)

$$x' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x$$

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

So,  $\lambda = 2$  is a second-order eigenvalue.

$$\begin{pmatrix} 3-1 \\ 1-1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} a-b=0 \\ a-b=0 \end{matrix} \text{ so, only 1 dimension of linearly independent eigenvectors.}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So, need to find a vector  $\vec{v}_2$  where:

$$(A-2I)^2 \vec{v}_2 = 0 \quad (A-2I) \vec{v}_2 = \vec{v}_1 \quad \vec{v}_1 \neq 0$$

$$A-2I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad (A-2I)^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, any  $\vec{v}_2$ , almost, will do. Take  $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\vec{v}_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$\vec{x}_2 = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{2t}$$

So, a fundamental matrix is:

$$\Phi(t) = \begin{pmatrix} e^{2t} & (t+1)e^{2t} \\ e^{2t} & te^{2t} \end{pmatrix}$$

6. Calculate the matrix exponential  $e^{\mathbf{A}t}$  for the matrix below. (Hint - check that the matrix is nilpotent): (5 points)

$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So,

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{pmatrix} 1+6t & 4t \\ -9t & 1-6t \end{pmatrix}$$



7. Find a ~~general~~ <sup>particular</sup> solution to the nonhomogenous system of ODEs:  
(5 points)

$$x' = 2x + y + 1$$

$$y' = 4x + 2y$$

~~First~~  $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Guess  $\vec{x}_p = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \vec{x}_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a+b+1 \\ 4a+2b \end{pmatrix}$  cannot work. This is because an eigenvalue  $\lambda=0$  exists for this matrix.

$$\vec{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\vec{x}_p' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \left[ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right] + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow a_1 = (2a_1 + a_2)t + (2b_1 + b_2) + 1$$

$$a_2 = (4a_1 + 2a_2)t + (4b_1 + 2b_2)$$

So, must have  $a_2 = -2a_1$

$$\Rightarrow \begin{aligned} a_1 &= (2b_1 + b_2) + 1 \\ -2a_1 &= 2(2b_1 + b_2) \end{aligned} \Rightarrow -4a_1 = -2 \Rightarrow a_1 = \frac{1}{2}$$

$$\text{and } (2b_1 + b_2) = -\frac{1}{2}$$

Note this is as far as we can go in solving this, as  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$  is a solution to the homogenous equation.

So, a particular solution would be:

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$$\boxed{x_p(t) = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}}$$

8. Please vote on which chapter you'd like to do after we finish chapter 7. We could either go back to systems of ODEs and do chapter 6, which deals with non-linear systems, or we could do chapter 8, which deals with power series methods. Unfortunately, we only have time for 1 (along with chapter 9) and so I figured I'd let the class decide. I'll choose to break any tie. (5 points)

- **Chapter 6** - Chapter 6 deals with nonlinear systems and phenomenon. Nonlinear systems are difficult to model, but they can be worthwhile because they frequently exhibit very interesting behavior and can be used to model many real-world phenomena for which linear systems are just not accurate. An old joke is that differential equations are divided between linear and nonlinear in the same way that living things are divided into zebras and non-zebras. In other words, linear phenomena is *everywhere*. There are many cool things that come out of nonlinear equations, like ecological models and the beginnings of chaos theory. However, as I said they're difficult to work with, and so this section will involve a fair number of in-depth calculations, along with quite a bit of graphing and picture-drawing, and a fairly meaty computer project.
- **Chapter 8** - Chapter 8 deals with power series methods. You probably remember power series from calculus II, well, here's where they come back with a vengeance. It turns out that using power series we can solve an enormous number of differential equations that would otherwise not be solvable using any of our known methods. Power series methods show up all the time in real world applications where you have to solve more difficult differential equations. The advantage to this chapter is that you learn a very powerful method that comes up all the time in applications, and the problems tend to be more straightforward and lend themselves to more exact calculations.

Anyways, as I said it's up to you. Please vote for the one you'd rather do. A vote for either chapter gives you the extra credit points. Should be the easiest points you earn all semester. Please put a check mark by the chapter you'd like to study, or do something else that indicates your preference.

Answers will, of course, vary.