Quiz 2 Review Notes

Math 2280

Spring 2008

Disclaimer - These notes are designed to be just that, notes, for a review of what we've studied since the first exam. They are certainly not exhaustive, but will provide an outline of the material that you will need to know for the second exam. They are intended as a study aid, but they should definitely not be the only thing you study. I also cannot guarantee they are free from typos, either in the exposition or in the equations. Please refer to the appropriate equations in the textbook while you're studying to make sure the equations in these notes are free of typos, and if you find a typo please contact me and let me know. If there are any sections upon which you're unclear or think you need more information, please read the appropriate section of the book or you lectures notes. If after doing this it's still unclear please ask me, either at my office, via email, or at the review session. If you think you need further practice in any area I'd recommend working the recommended problems in the textbook from the sections that cover the material on which you're having trouble.

1 Mechanical Systems Modeled by Linear Equations of Higher Order

We finished off chapter 3 by studying how higher order linear ODEs are applied to mechanical and electrical systems in the real world.

1.1 The Mass-Spring-Dashpot System

The mechanical system that we studied the most was the mass-spring-dashpot system pictured below:

Mass-Spring-Dashpot System

with mass m, damping constant c, and spring constant k. The ODE for this system is:

$$mx''(t) + cx'(t) + kx(t) = 0$$

where $x(t)$ is the mass's displacement from its equilibrium position. Now, if there is an external driving force $F(t)$ (pictured schematically in the picture as the arrow) then we get the potentially more complicated ODE:

$$mx''(t) + cx'(t) + kx(t) = F(t).$$

Now, the most commonly studied situation here is when the driving force is a periodic sinusoidal function $F(t) = F_0 \cos{(\omega t - \alpha)}$, where F_0 is the amplitude, ω is the angular fequency, and α is the phase, which determines the initial positon. If this is the case then our methods for calculating particular solutions, namely the method of undetermined coefficients, works to solve problems of this type.

If the damping constant c is not present, that is $c = 0$, then the mass-spring system oscillates back and forth with an angular frequency ω_0 given by:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

if the driving function $F(t) = F_0 \cos(\omega t - \alpha)$ has the same frequency as the natural frequency of vibration for the mass-spring system, $\omega_0 = \omega$, then we get the phenomenon of resonance. In this case our solution $x(t)$ will have a factor of the form $At \cos(\omega_0 t - \beta)$ which, because of the t term multiplying the cosine, makes the amplitude of the oscillations grow without bound as time increases.

1.2 Electrical Circuits

An electrical circuit of the form below:
Inductor-Resistor-Capacitor System

has a mathematical model that is fundamentally the same as the model used for our mass-spring-dashpot system with a driving force. Namely, we have the ODE:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

So, all the mathematical methods and results from our mechanical system carry over into our electrical system. Note here that Q represents the amount of charge on the capacitor. The movement of charge causes current, so $\frac{dQ}{dt} = I$, where I is the current in our circuit. Frequently we work with current, as in practice it's almost always easier to measure, and so if we take derivatives our equation above becomes:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t).$$

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Note that removing any circuit element in the above schematic is equivalent to setting the coefficient for that circuit (L, R, or C) in the above ODEs equal to 0. So, using ODE methods we already know we should be able to solve circuits like the one above both when all the circuit components are there, and when we remove one or two of them.

2   Systems of Differential Equations

In chapters 4 and 5 we studied systems of ordinary differential equations and methods for solving them.

2.1   Systems of Differential Equations

A system of differential equations is a set of coupled differential equations. So, it's a system where some order derivative of one variable depends not just upon its own values, but also on the values of otehr variables and their derivatives.

So, for example, we could have the second order system:

$$m_1 x'' = -k_1 x + k_2(y - x)$$
$$m_2 y'' = -k_2(y - x) + f(t)$$

where the values of x'' and y'', and consequently the values of x and y, depend on the value of the other respective variable.

2.2   First Order Systems

A first order system of ODEs is a system in which only first derivatives of the variables appear. So, for example, the system given above would not be a first order system, while the superficially similar system:

$$m_1 x' = -k_1 x + k_2(y - x)$$
$$m_2 y' = -k_2(y - x) + f(t)$$

would be an example of a first order system in that on the left hand side we only find first derivatives of our variables, while on the right hand side we only find the variables with no derivatives.

We note something important, in that *any* system of ODEs can be turned into a first-order system of ODEs with the introduction of more variables. Explaining how this is done in generality is tricky and more confusing than it's worth, as the method itself is very easy. So, I'll just give an example of how it would be done by converting the second-order ODE at the beginning of this section into a first-order ODE.

Introduce the variables $y_1 = y'$ and $x_1 = x'$. Then our system of two second-order ODEs:

$$m_1 x'' = -k_1 x + k_2(y - x)$$
$$m_2 y'' = -k_2(y - x) + f(t)$$

becomes the system of four first-order ODEs:

$$m_1 x_1' = -k_1 x + k_2(y - x)$$
$$m_2 y_1' = -k_2(y - x) + f(t)$$
$$y' = y_1$$
$$x' = x_1.$$

The important thing here is that, by just focusing on systems of first-order ODEs, there's really no loss in generality, as the methods can be, in principal, applied to systems of any order with the appropriate introduciton of new variables and new equations.

2.3 Linear First-Order Systems

We will restrict our attention pretty exclusively to linear first-order systems, which are systems of the form:

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t) \\
&\ \vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t)
\end{aligned}
$$

where all the terms $p_{ij}(t)$ and $f_i(t)$ are continuous functions on some time interval.

Now, for linear systems in the form above we have the following theorem:

Suppose that the funcitons $p_{11}, p_{12}, \ldots, p_{nn}$ and the functions $f_1, f_2, \ldots, f_n$ are continuous on the open interval I containing the point a. Then, given the n numbers $b_1, b_2, \ldots, b_n$ the system above has a unique solution on the entire interval I that satisfies the n initial conditions:

$$x_1(a) = b_1, \, x_2(a) = b_2, \, \ldots, \, x_n(a) = b_n.$$

So, we have another useful existance and uniqueness theorem. Note that this is actually equivalent to the existance and uniqueness theorem for higher-order linear ODEs that we learned in chapter 3.

2.4   Solving Systems of ODEs by Elimination

We discussed earlier how higher-order systems of ODEs can be converted into first-order systems by introducing more equations and variables. Well, frequently we can go the other way and take a system of first-order ODEs and reduce it to some higher-order ODE in one variable, and then use our methods from chapter 3 to solve it. There's nothing too tricky about it, and so it's probably best illustrated with an example.

Example
Solve the system of ODEs:

$$x' = -3x + 2y$$
$$y' = -3x + 4y$$

with initial conditions $x(0) = 0$ and $y(0) = 2$.

If we take the derivative of the first equation we get:

$$x'' = -3x' + 2y'$$

and if we substitute the relation in the second equation for y' we get:

$$x'' = -3x' + 2(-3x + 4y) = -3x' - 6x + 8y$$

and now if we use the relation from the first equation $2y = x' + 3x$ we get:

$$x'' = -3x' - 6x + 4(x' + 3x) = x' + 6x$$

and so we get the second-order ODE for $x(t)$:

$$x''(t) - x'(t) - 6x(t) = 0$$

which we can solve using methods from chapter 3 to get:

$$x(t) = c_1 e^{-2t} + c_2 e^{3t}$$

and so if we solve for y using our earlier relation $2y = x' + 3x$ we get:

$$2y = -2c_1 e^{-2t} + 3c_2 e^{3t} + 3c_1 e^{-2t} + 3c_2 e^{3t} = c_1 e^{-2t} + 6c_2 e^{3t}.$$

Now, using our initial conditions we get:

$$c_1 + c_2 = 0$$
$$\text{and}$$
$$c_1 + 6c_2 = 4$$

which when solved gives us:

$$c_1 = -\frac{4}{5} \text{ and } c_2 = \frac{4}{5}$$

and therefore our final answer is:

$$x(t) = -\frac{4}{5}e^{-2t} + \frac{4}{5}e^{3t}$$
$$\text{and}$$
$$y(t) = -\frac{2}{5}e^{-2t} + \frac{12}{5}e^{3t}$$

So, as you can see, eliminating and solving can be kind of a pain, but it's not that difficult conceptually.

3   Linear Systems and Linear Methods

I won't review linear algebra here (the basics are covered in section 5.1 if you need a review) except to say that all the concepts we know from linear algebra and matrices can be applied to situations where instead of insisting that the entries in the matrices be scalars, we allow them to be functions.

So, we can have matrix valued functions:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

and we can represent first-order linear systems as products and sums of matrix-value functions:

$$\begin{pmatrix} x'_1(t) \\ x'_2(t) \\ \vdots \\ x'_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

which can be written *much* more compactly as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$$

Note that if our system of first-order linear equations has $\mathbf{f} = 0$ then it is called homogeneous. If $\mathbf{f} \neq 0$ then it is non-homogeneous.

3.1   Theory of Linear Systems

The theory behind, and important theorems in, the study of systems of linear ODEs is very similar to the theory and theorems in the study of higher-order linear ODEs. The big theorems are:

Principle of Superposition
If $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$ are n solutions of the homogenous linear equation above on the open interval I. If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n(t)$$

is also a solution of the homogenous linear equation on the interval I.

Wronskian of Solutions

Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$ are n solutions of the homogenous linear equation $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}$ on an open interval I. Suppose also that $\mathbf{P}(t)$ is continuous on I. Let

$$W = W(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n).$$

be the determinant of the matrix formed by using the n solutions as columns. This is called the Wronskian.

Then:

- If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly dependent on I, then $W = 0$ at every point of I.

- If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent on I, then $W \neq 0$ at each point of I.

  Thus there are only two possibilities for solutions of homogenous systems. Either $W = 0$ at *every* point of I, or $W = 0$ at *no* point of I.

General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I, where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of teh equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

for all t in I.

What this theorem basically says is that once you've found n linearly independent solutions you've essentially found all the solutions, as any other solution can be built as a linear combination of the n linearly independent solutions you've found so far.

Solutions of Nonhomogeneous Systems

Let $\mathbf{x}_p$ be a particular solution of a nonhomogeneous linear equation on an open interval I on which the functions $\mathbf{P}(t)$ and $\mathbf{f}(t)$ are continuous. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be linearly independent solutions of the associated homogeneous equation on I. If $\mathbf{x}(t)$ is any solution whatsoever of the homogeneous equation on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t)$$

for all t in I.

3.2   The Eigenvalue Method

The eigenvalue method is used for solving systems of ODEs with constant coefficients. The basic idea is that if you have a linear first-order system $\mathbf{x}' = \mathbf{Ax}$ and the matrix $\mathbf{A}$ is such that all the entries are constants, then solving the system is about as easy as finding the eigenvalues for the matrix $\mathbf{A}$ and the associated eigenvectors.

The idea is this. Suppose that a solution to the system $\mathbf{x}$ has the form $\mathbf{v}e^{\lambda t}$ where $\mathbf{v}$ is a non-zero constant vector. Then this would imply that $\mathbf{x}$ satisfied the relation:

$$\mathbf{x}' = \lambda \mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t}$$

which reduces to

$$\lambda \mathbf{v} = \mathbf{A}\mathbf{v}.$$

So, the solution will work if we can find a number λ and a non-zero vector $\mathbf{v}$ that satisfy the above relation. This just means that λ must be an eigenvalue of the matrix $\mathbf{A}$, and $\mathbf{v}$ must be its associated eigenvector.

Now, if all the eigenvalues are distinct and real, then we're done. There will be n such eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$ and our solution to the system of ODEs will be:

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}.$$

Now, if the eigenvalues are complex then they always come in pairs, namely the eigenvalue and its complex conjugate. Similarly, the eigenvector for the conjugate will just be the conjugate of the eigenvector for

the original eigenvalue. Read it twice, it does make sense. Anyways, the eigenvector for a complex eigenvalue may itself be convex, and so if our eigenvalue is of the form $\lambda = p + qi$ then our eigenvector will be of the form $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ and if we run through the math we get that the two solutions these eigenvalues generate are:

$$e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt) \text{ and } e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)$$

So, that's basically how you solve any homogenous linear system of first-order equations. The only problem is what to do when you have repeated roots.

Now, there are general methods for finding solutions when the eigenvalues are repeated, but beyond a second-order repeated root they become a major pain, so you only need to worry about how to handle second order eigenvalues. In this case there are two possibilities. First, the eigenvalue may have two linearly independent eigenvectors, $\mathbf{v}_1$, $\mathbf{v}_2$. If this is the case then there are no problems, as the two solutions $\mathbf{v}_1 e^{\lambda t}$ and $\mathbf{v}_2 e^{\lambda t}$ are two linearly independent solutions.

On the other hand, if there is only one linearly independent eigenvector for the given eigenvalue, things are a little bit more tricky. You must first fine a nonzero solution to the equation:

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = 0$$

such that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$$

where $\mathbf{v}_1$ is an eigenvector for the eigenvalue λ. Then the two solutions associated with the eigenvalue λ are:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}$$
$$\text{and}$$
$$\mathbf{x}_2(t) - (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t}$$

So, that's pretty much how you handle homogenous first-order linear systems with constant coefficients. Definitely go over this and review this in your book because you will need to know this stuff for the exam.

3.3 Matrix Exponentials

If we have n linearly independent solutions solutions to the n first-order homogenous system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ then we can form a matrix whose columns are the linearly independent solutions. (Note that this would be the same matrix whose determinant we'd take in order to find the Wronskian.) Now, by the properties of linear algebra, we have that given every column satisfies the system of differential equations, the entire matrix satisfies the relation:

$$\Phi'(t) = \mathbf{A}\Phi(t)$$

where $\Phi(t)$ is the matrix of solution vectors, called the fundamental matrix. Now, if we have the (or I should say "a" fundamental matrix, as there will be many for any given system) then we can solve any initial value problem $\mathbf{x}(0) = \mathbf{x}_0$ for the solution $\mathbf{x}(t)$ that coincides with that initial value. Precisely, the solution $\mathbf{x}(t)$ is given by:

$$\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$$

Now, these fundamental matrices can be found using eigenvalue methods, but they can also be found using matrix exponentials. A matrix exponential is defined as:

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}$$

Now, the important property of matrix exponentials is that if $\Phi(t) = e^{\mathbf{A}t}$ then $\Phi'(t) = \mathbf{A}e^{\mathbf{A}t}$. So, $e^{\mathbf{A}t}$ is a fundamental matrix for the homogenous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and so

For the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{x}(0) = \mathbf{x}_0$ the solution is given by:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0.$$

We note that we can calculate any matrix exponential from another fundamental matrix using the formula:

$$e^{\mathbf{A}t} = \Phi(t)\Phi(0)^{-1}.$$

3.4 Nonhomogenous Linear Systems

The final question confronted in chapter 5 is how do we solve *nonhomogenous* first-order linear systems with constant coefficient matrices. Well, as mentioned earlier, the solution will be a combination of the homogenous solution and a particular solution, so the question is how to find this particular solution.

Well, it turns out the methods for solving nonhomogenous linear ODEs transfer pretty directly into solving nonhomogenous linear systems of ODEs. Namely, there is a method of undetermined coefficients and a method of variation of parameters, although here the method of variation of parameters can be much more useful.

Method of Undetermined Coefficients

This is pretty much the same as the method of undetermined coefficients from linear ODEs. So, for example, if you were given the system:

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & 5 \\ 2 & 6 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sin t \\ t^2 \end{pmatrix}$$

You'd break up the nonhomogenous part $\mathbf{f}(t)$ as:

$$\mathbf{f}(t) = \begin{pmatrix} \sin t \\ t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^2$$

and guess that our particular solution has the form:

$$\mathbf{x}_p = \mathbf{v}_1 \sin t + \mathbf{v}_2 t^2 + \mathbf{v}_3 t + \mathbf{v}_4$$

where the $\mathbf{v}_i$ terms would be determined by the particular equation. But, just find the appropriate vectors $\mathbf{v}_i$ and the particular solution is found.

The only caveat that makes this method really different from the method used for nonhomogenous linear ODEs is that if one of the terms in the "guess" is linearly dependent upon the homogenous solution you don't just multiply it and its similar terms by t. You do this, but you also must include the lower order terms. So, for example, if we had the nonhomogenous first-order system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$$

where

$$\mathbf{x} = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

then we'd guess that our particular solution would have the form:

$$\mathbf{x}_p = \mathbf{v}_1 t + \mathbf{v}_0$$

However, if $\mathbf{v}_0$ is not linearly independent from our homogenous solution we cannot replace our guess with:

$$\mathbf{x}_p = \mathbf{v}_1 t^2 + \mathbf{v}_1 t$$

we must instead replace it with:

$$\mathbf{x}_p = \mathbf{v}_2 t^2 + \mathbf{v}_1 t + \mathbf{v}_2.$$

So, we must introduce "higher order" terms but we cannot discard the "lower order" terms when we do this.

Variation of Parameters

The idea, theory, and derivation of the variation of parameters method can be found at the end of section 5.6 of the textbook. The cut to the chase important result is this:

If $\Phi(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on some interval where $\mathbf{P}(t)$ and $\mathbf{f}(t)$ are continuous, then a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

is given by

$$\mathbf{x}_p = \Phi(t) \int \Phi(t)^{-1} \mathbf{f}(t) dt.$$

where $Phi(t)$ is a fundamental matrix for the homogenous system of ODEs. Note that in order to use this method you must first solve the attendant homogenous equation and find n linearly independent solutions.

Good Luck on the Exam!