Math 2280 - Quiz 1

University of Utah

Spring 2008

Name: Solutions

1. (1 point) What is the order of the differential equation in terms of the function y:

$$y^{(3)} \sin 5x - (y'')^2 y' + e^{5xy} + \log y' + y^{(4)} x = \frac{\tan x}{x^2 + 2x + 1}$$

4

2. (4 points)

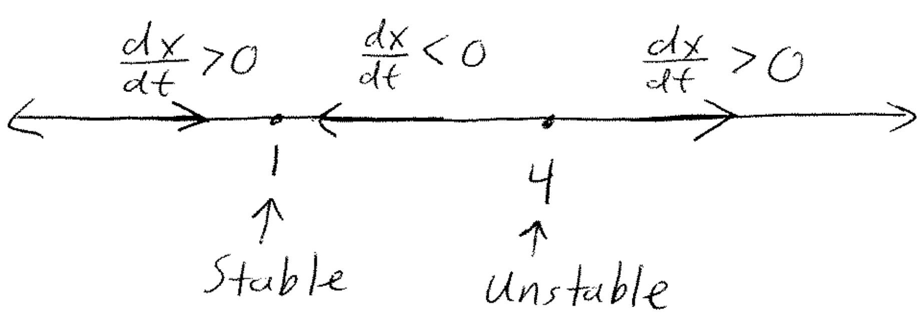
Find the critical points of the given autonomous differential equation. Then, draw the phase diagram for the differential equation and determine for each critical point if it is stable or unstable.

$$\frac{dx}{dt} = x^2 - 5x + 4$$

$\frac{dx}{dt} = (x-4)(x-1) = 0$ when $x = 1$ or 4

So, critical points at $x = \{1, 4\}$.

Phase Diagram



For problems 3-5 solve the given ODE. If initial conditions are specified, find the solution that matches the initial conditions. If no initial conditions are specified, find the general form of the solution.

3. (5 points)

$$\frac{dy}{dx} = 4x^3y - y \text{ and } y(1) = -3$$

$\frac{dy}{dx} = (4x^3-1)y \Rightarrow \int \frac{dy}{y} = \int (4x^3-1)dx$

$\Rightarrow \ln|y| = x^4 - x + C \Rightarrow y = Ce^{x^4-x}$

$y(1) = Ce^{1^4-1} = Ce^0 = C = -3$

So, $\boxed{y(x) = -3e^{x^4-x}}$

4. (5 points)

$$(1+x)y' + y = \cos x \text{ and } y(0) = 1$$

$\Rightarrow y' + \frac{y}{1+x} = \frac{\cos x}{1+x}$ assuming $x \neq -1$

$\rho(x) = e^{\int \frac{1}{1+x}dx} = e^{\ln|1+x|} = 1+x$

So,

$\frac{d}{dx}((1+x)y) = \cos x$

$\Rightarrow \cancel{4}(1+x)y = \int \cos x = \sin x + C$

$\Rightarrow y = \frac{\sin x}{1+x} + \frac{C}{1+x}$ at $x=0$ $y = 0 + C = 1$

$\Rightarrow \boxed{y(x) = \frac{\sin x + 1}{1+x}}$

5. (5 points) *Note* - You may express the solution here implicitly.

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$$

Check it's exact:

$$\frac{\partial(1+ye^{xy})}{\partial y} = xye^{xy} + e^{xy}$$

$$\frac{\partial(2y+xe^{xy})}{\partial x} = xye^{xy} + e^{xy}$$

So, it's exact, and

$$\frac{\partial F}{\partial x} = 1+ye^{xy} \Rightarrow F = \int (1+ye^{xy})dx$$

$$= x + e^{xy} + g(y)$$

$$\frac{\partial F}{\partial y} = xe^{xy} + g'(y) = 2y + xe^{xy}$$

$$\Rightarrow g'(y) = 2y \Rightarrow g(y) = \int 2y = y^2$$

So,

$$F = \boxed{x + e^{xy} + y^2 = C}$$

Defines our solution implicitly.

6. (5 points) Solve for the general form of the function $y(x)$.

$$9y^{(3)} + 12y'' + 4y' = 0$$

Homogenous linear ODE so the characteristic equation is: $9r^3 + 12r^2 + 4r = 0$

$\Rightarrow r(9r^2+12r+4) = r(3r+2)(3r+2) = r(3r+2)^2$

So, solution is $r = \{0, -\frac{2}{3}\}$ with $-\frac{2}{3}$ of order 2

$\Rightarrow \boxed{y(x) = c_1 + c_2 e^{-2/3 x} + c_3 x e^{-2/3 x}}$

7. (5 points) Find a particular solution y_p to the ODE:

$$y^{(4)} - 4y'' = x^2.$$

First the homogenous solution is:

characteristic polynomial:

$r^4 - 4r^2 = 0 \Rightarrow r^2(r-2)(r+2)$

So, homogenous solution is:

$y_h = c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x}$

First guess $\longrightarrow y_p = Ax^2 + \underbrace{Bx + C}_{\text{not linearly independent}}$

So, need

$y_p = Ax^4 + Bx^3 + Cx^2 \qquad y_p^{(4)} = 24A$

$y_p'' = 12Ax^2 + 6Bx + 2C$

$\Rightarrow 24A - 4(12Ax^2 + 6Bx + 2C) = x^2$

$\Rightarrow A = -\frac{1}{48} \quad B = 0 \quad C = -\frac{1}{16} \quad \Rightarrow \boxed{y_p = -\frac{1}{48}x^4 - \frac{1}{16}x^2}$

8. (5 points) Are we guaranteed existance and uniqueness of the given differential equation in a neighborhood of the given point? Why or why not?

$$\frac{dy}{dx} = \sqrt{x-y};\ y(2) = 1$$

$f(x,y) = \sqrt{x-y} = 1$

$\frac{\partial f}{\partial y} = \frac{-1}{2\sqrt{x-y}} = -\frac{1}{2}$ at $(2,1)$. Both continuous around $x-y>0$.

Both continuous at $(2,1)$ so a unique solution exists.

9. (5 points) Suppose that a body moves through a resisting medium with resistance proportional to its velocity v, so that $\frac{dv}{dt} = -kv$. Show that its velocity and position at time t are given by:

$$v(t) = v_0 e^{-kt}$$

$$x(t) = x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt}).$$

Where v_0 and x_0 are the respective velocity and position at time $t = 0$.

$\int \frac{dv}{v} = \int -k\,dt \Rightarrow \ln v = -kt + C$

$\Rightarrow v(t) = Ce^{-kt}$

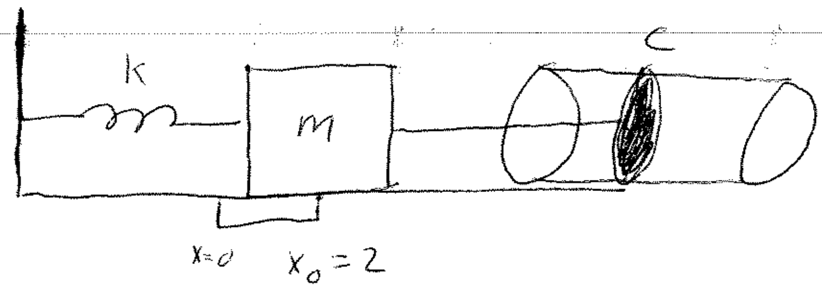
$v(0) = v_0 = C \Rightarrow \boxed{v(t) = v_0 e^{-kt}}$

$x(t) = \int v(t) = -\frac{v_0}{k} e^{-kt} + C$

$x(0) = x_0 = C - \frac{v_0}{k} \Rightarrow C = x_0 + \frac{v_0}{k}$

So, $x(t) = -\frac{v_0}{k} e^{-kt} + x_0 + \frac{v_0}{k} = \boxed{x_0 + \frac{v_0}{k}(1 - e^{-kt})}$

10. (5 points) For the mechanical system pictured below with $m = 2$, $c = 3$ and $k = 1$ what is the equation $x(t)$ that describes the motion of the mass around its equilibrium point if $x(0) = 2$ and $x'(0) = 0$? Is the system overdamped, underdamped, or critically damped?



$$mx'' + cx' + kx = 0$$

$$2x'' + 3x' + x = 0$$

characteristic equation

$$2r^2 + 3r + 1 = 0$$

roots $r = \dfrac{-3 \pm \sqrt{3^2 - 4(2)(1)}}{2(2)}$

$= \dfrac{-3 \pm 1}{4} = \{-1, -\frac{1}{2}\}$

So,

$x(t) = c_1 e^{-x} + c_2 e^{-\frac{x}{2}}$ $\qquad X(0) = c_1 + c_2 = 2$

$x'(t) = -c_1 e^{-x} - \frac{c_2}{2} e^{-x/2}$ $\qquad x'(0) = -c_1 - \frac{c_2}{2} = 0$

$\Rightarrow \frac{c_2}{2} = 2 \Rightarrow c_2 = 4$

$c_1 = -2$

So,

$$\boxed{X(t) = -2e^{-x} + 4e^{-x/2}}$$

Overdamped

11. (5 points) Use Euler's method to fill in the rest of the table below for the differential equation below with the given initial conditions and step size. *Note* - The y_n in the table below represent your estimated values given by Euler's method.

$$y' = 2xy^2;$$

$$y(0) = 1;$$

$$h = .5.$$

$$y_1 = y_0 + h\, y'(x_0, y_0)$$
$$= 1 + .5\,(2(0)(1^2))$$
$$= 1$$

n	x	y_n
0	0	1
1	.5	1
2	1	1.5
3	1.5	~~3.25~~ 3.75

$$y_2 = 1 + .5\,(2(.5)(1^2))$$
$$= 1 + .5$$
$$= 1.5$$

$$y_3 = 1.5 + .5\,(2(1)(1.5)^2)$$
$$= \frac{3}{2} + \frac{1}{2}\left(2\left(\frac{9}{4}\right)\right)$$
$$= \frac{3}{2} + \frac{9}{4}$$
$$= \cancel{\frac{13}{4} = 3.25} \quad \frac{15}{4} = 3.75$$

12. (5 points) Extra Credit.

In the derivation of the variation of parameters method for second-order linear ODEs we learned that our particular solution will be of the form:

$$y_p = u_1 y_1 + u_2 y_2$$

where y_1 and y_2 are two linearly independent solutions to the corresponding homogenous solution, and u_1 and u_2 are undetermined functions. We derived that the functions u_1 and u_2 would have to satisfy the form:

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = f(x).$$

We solved this system to get:

$$u_1 = -\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx; \text{ and } u_2 = \int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx.$$

where W represents the Wronskian. Explain how we get from the system of two equations above to the given solutions. *Note* - Write on the back of the page if you run out of room on this one.

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

$$\Rightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

Inverting the matrix $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ we get:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} = \frac{1}{W(y_1, y_2)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W(y_1, y_2)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

$$\Rightarrow u_1' = \frac{-y_2 f}{W(y_1, y_2)} \qquad u_2' = \frac{y_1 f}{W(y_1, y_2)} \quad \star$$

So,

$$y_p = u_1 y_1 + u_2 y_2 = -y_1 \int \frac{y_2 f}{W(y_1, y_2)} + y_2 \int \frac{y_1 f}{W(y_1, y_2)}$$

where given $\star$ we have:

$$u_1 = \int \frac{-y_2 f}{W(y_1, y_2)} \qquad u_2 = \int \frac{y_1 f}{W(y_1, y_2)}$$