Quiz 1 Review Notes

Math 2280

Spring 2008

Disclaimer - These notes are designed to be just that, notes, for a review of what we've studied thus far in class. They are certainly not exhaustive, but will provide an outline of the material that you will need to know for the first quiz. They are intended as a study aid, but they should definitely not be the only thing you study. I also cannot guarantee they are free from typos, either in the exposition or in the equations. Please refer to the appropriate equations in the textbook while you're studying to make sure the equations in these notes are free of typos, and if you find a typo please contact me and let me know. If there are any sections upon which you're unclear or think you need more information, please read the appropriate section of the book or you lectures notes. If after doing this it's still unclear please ask me, either at my office, via email, or at the review session. If you think you need further practice in any area I'd recommend working the recommended problems in the textbook from the sections that cover the material on which you're having trouble.

1 The Basics

An ordinary differential equation is an equation indicating a relation between a function of one variable and its derivatives. For example:

$$(y'')^2 + 3x = y$$

or

$$4y^{(3)} + 2y = 5.$$

The goal in differential equations is to find the function that satisfies the given ordinary differential equation. Frequently there are whole families of functions that will satisfy a given differential equation, and so to pick out a particular solution initial conditions are specified. The initial conditions are usually the value of the function and a certain number of the functions derivatives at a given point of interest.

1.1 Order

The order of a ordinary differential equation is the maximum derivative that appears in the defining equation. So, for example, in the two example ODEs given above the orders are 2 and 3, respectively.

1.2 Linear ODEs

A linear ODE is an ODE in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y^{'} + a_0(x)y = f(x)$$

where the $a_i(x)$ and $f(x)$ represent functions that are all continuous on some interval of interest. We usually assume that the function $a_n(x)$ is also non-zero on this interval of interest, allowing us to divide both sides of the equation by it to get the equivalent "monic" equation:

$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + \cdots + b_1(x)y^{'} + b_0(x)y = g(x)$$
$$\text{where } b_i(x) = \frac{a_i(x)}{a_n(x)} \text{ and } g(x) = \frac{f(x)}{a_n(x)}.$$

We call a linear ODE *homogenous* if $f(x) = 0$ and we call is a linear ODE with constant coefficients if all the functions $a_i(x)$ are constants.

2 Theory

There are a couple of big existance and uniqueness theorems that you should know how to apply, if not how to prove. There are:

- Suppose that both the function $f(x, y)$ and its partial derivative $\frac{\partial f}{\partial y}$ are continuous on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x,y),\ y(a) = b$$

has one and only one solution that is defined on some interval I that contains the point x. (Note that this interval may not be as wide as the width of the rectangle R. I could have a longer or shorter width than R)

- Suppose that the functions $a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ and f are continuous on the open interval I containing the point a. Then, given n numbers $b_0, b_1, \ldots, b_{n-1}$ the nth-order linear equation:

  $$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots a_1(x)y' + a_0(x)y = f(x)$$

  has a unique (one and only one) solution on the entire interval I that satisfies the n initial conditions

  $$y(a) = b_0, y'(a) = b_1, \ldots, y^{(n-1)}(a) = b_{n-1}$$

For an example of how the first theorem is applied check out example 6 from section 1.3 of the textbook. For an example of the second theorem for the special case of a first order linear ODE check out example 3 from section 1.5 of the text. For an example in the more general context check out examples 1 and 2 from section 3.2 of the text.

3 First-Order ODEs

We've learned some techniques for solving certain types of first order ODEs. You should know these methods and how to use them for the quiz.

3.1 Separable First-Order ODEs

A separable first-order ODE is an ODE that can be written in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

if this is the case then we can solve for y by rearranging and then integrating the above equation:

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

and then solving for y.

For example, the ODE:

$$\frac{dy}{dx} = y$$

is separable with $f(x) = 1$ and $g(y) = y$. Separating this and integrating we get:

$$\begin{aligned} \int \frac{dy}{y} &= \int dx \\ \to \ln|y| &= x + C \\ \to y &= Ce^x \end{aligned}$$

where the two constants C above would be different, but we're using the convention of just reusing the same letter when you turn one unknown constant into another, still unknown, constant. There are many more separable and more complicated ODEs. These can be found in section 1.4 of the text.

3.2 Linear First-Order Equations

A linear first-order equation is an equation of the form:

$$\frac{dy}{dx} + p(x)y = f(x).$$

Any equation of this form can be solved by multiplying both sides by the integrating factor $\rho(x) = e^{\int p(x)dx}$ to get:

$$\begin{aligned} \rho(x)\frac{dy}{dx} + \rho(x)p(x)y &= \rho(x)f(x) \\ \to \frac{d}{dx}(\rho(x)y) &= \rho(x)f(x) \end{aligned}$$

and so:

$$y = \frac{1}{\rho(x)} \int \rho(x)f(x)dx$$

So, for example, if we're given the ODE:

$$xy' + 3y = x^3$$

we can rearrange this to form:

$$y' + \frac{3}{x}y = x^2$$

Note that this breaks our solution up into two intervals, one for $x > 0$ and one for $x < 0$, because these are the intervals on which the division above doesn't cause problems (doesn't lead to division by 0). Solving this ODE we get:

$$\rho(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

and so

$$y = \frac{1}{x^3}\int x^5 dx = \frac{x^6}{6x^3} + \frac{C}{x^3} = \frac{x^3}{6} + \frac{C}{x^3}$$

Now, if we're given the initial condition $y(1) = 10$ we get:

$$y(1) = \frac{1}{6} + C = 10$$

and so

$$C = 10 - \frac{1}{6} = \frac{59}{6}$$

which gives us a final solution:

$$y(x) = \frac{x^3}{6} + \frac{59}{6x^3}$$

We note that this solution will be unique for this differential equation and this initial conditions on the interval $x > 0$.

More problems and further explanation of this method can be found in section 1.5 of the text.

3.3 Substitution Methods

Frequently we can turn a differential equation that we don't immediately know how to solve into a more familiar differential equation that we do know how to solve by a substitution. This method is analogous to u substitution in single variable calculus. Basically, this method involves making a substitution of the form $v = \alpha(x, y)$. There are no absolute rules for using this method, and sometimes it will work and sometimes it won't. There are a few particular situations where this method is particularly useful, and so you should know these.

3.3.1 Homogenous Equations

A *homogenous* first-order differential equation is one that can be written in the form:

$$\frac{dy}{dx} = F(\frac{y}{x}).$$

If this is the case then we can make the substitution $v = \frac{y}{x}$, which gives us the corresponding relations:

$$y = vx \text{ and } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

and we transform the above equation into a separable equation:

$$x\frac{dv}{dx} = F(v) - v$$

which we can then solve for v, and after that we can solve for y.

3.3.2 Bernoulli Equations

A first-order differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a Bernoulli equation. If $n = 0$ or $n = 1$ we can solve this using previous methods. If $n > 1$ then we can make the substitution $v = y^{1-n}$, which turns the equation into the linear equation:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

which we then can solve using previously covered methods.

3.3.3 Reducible Second Order Equations

If we have a second-order ODE:

$$F(x, y, y', y'') = 0$$

where the independent variable y is missing from the equation, then the substitution $p = y'$, and hence $p' = y''$ will turn the equation into a first-order ODE, which we can then solve for p. From p we can integrate to get y.

If the independent variable x is missing then if we make the substitution $p = y'$ we get the relation:

$$p = y', y'' = p\frac{dp}{dy}$$

and our equation becomes an equation of the form:

$$F(y, p, p\frac{dp}{dy}) = 0$$

which is a first order equation that we can solve for p, and from p we can solve for x, and from x we can, perhaps, solve for y.

3.4 Exact Equations

An exact ODE is a first order differential equation that can be written in the form:

$$M(x, y)dx + N(x, y)dy = 0$$

where the function M and N satisfy the relations:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If this is the case then there is an implicit solution F to the ODE that satisfies:

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N.$$

For an example of how this is done check out example 9 of section 1.6 from the text. In general for more about substitution methods and exact ODEs check out section 1.6 from the text.

4 Further Topics in First-Order ODEs

Here we'll take a look at a couple of further topics in ODEs that we investigated.

4.1 Equilibrium Solutions and Stability

If we have a first-order ODE of the form:

$$\frac{dy}{dx} = f(y)$$

then any of the "roots" of the function $f(y)$, in other words the points c where $f(c) = 0$, present particularly simple solutions. Namely, if we're given the initial condition $y(0) = c$ then the constant solution $y(x) = c$ satisfies the differential equation! This is called an equilibrium solution to the ODE, and frequently in equations of this type the other solutions will, as x increases, either go off to $\pm\infty$ or approach one of the equilibrium solutions.

Now, these equilibrium solutions can be either stable or unstable. A stable solution means, basically, that if you move away from it a slight amount in any direction you'll return back to the solution. An unstable solution is one where if you move away in one or both directions you fall away and don't come back.

For example, if we have the first-order ODE:

$$\frac{dy}{dx} = ky(y - M)$$

this equation has two equilibrium solutions, $y = 0$ and $y = M$. If we solve this ODE we get the solution:

$$y(x) = \frac{My_0}{y_0 + (M - y_0)e^{kMt}}$$

Now, if we graph this solutions for different values of the initial condition y_0 we get solutions curves that look like:

We can see that around the equilibrium point M we have an unstable equilibrium, while around the critical point 0 we have a stable equilibrium. This can be represented schematically by something called a *phase diagram*. This is a one-dimensional diagram where values on the axis represent different values of the variable y. We mark off where the equilibrium points are, and then in between these points we indicate what the sign of the derivative $\frac{dy}{dx}$ is. We draw arrows to the right if the sign is positive, and arrows to the left is the sign is negative. Any equilibrium point with two arrows pointing into it is a stable equilibrium point. Any other equilibrium point is an unstable equilibrium point. For example, the phase diagram for our above differential equation would be:

which tells us schematically what the earlier picture also told us. Namely, that 0 is a stable equilibrium point, while M is an unstable equilibrium point.

4.2 Euler's Method

Euler's method is the first numerical method we've learned so far, and it's pretty intuitive and easy to use. Basically, the idea is that you're given some first order ODE in the form:

$$\frac{dy}{dx} = f(x, y)$$

and an initial value $y(x_0) = y_0$. From this initial condition we estimate further values of the function $y(x)$ using the algorithm:

$$\tilde{y}(x + h) = \tilde{y}(x) + h \times f(x, \tilde{y}(x))$$

where $\tilde{y}$ is out estimate of y, $\tilde{y}(x_0) = y_0$ is our starting value, and h is a parameter called our step size. In general, the approximations $\tilde{y}$ will be closer to the actual function y the smaller the step size h is. For a description of the motivation for this algorithm check out section 2.4 of the text. For an example of how it's applied check out example 1 from that section.

5 Higher-Order ODEs

We've primarily dealt with linear higher-order ODEs, as they're by far the most tractable of the higher-order ODEs. They're the ones you'll need to know how to solve for the quiz.

5.1 Basic Concepts

For a linear ODE of the form:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots a_1(x)y' + a_0(x)y = f(x)$$

we call the associated equation:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots a_1(x)y' + a_0(x)y = 0$$

the associated *homogenous* linear ODE. For a n-th order linear ODE the associated homogenous ODE will have n linearly independent solutions $\{y_1, y_2, y_3, \ldots, y_n\}$ and so any solution to the homogenous equation can be written as:

$$y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where the coefficients c_i would be determined by the n initial conditions of the system.

The final solution to the linear ODE will be the sum of the homogenous solution and a particular solution y_p. The particular solution will be a function that actually solves the full ODE. So, the final solution y will be of the form:

$$y = y_p + y_h$$

where, as mentioned before, whatever initial conditions there are will be handled by assigning different values to the the various coefficients of the homogenous solution.

5.2 Linear Independence

We mentioned above that the homogenous solution will consist of the linear combination of n linearly independent solutions to the homogenous equation. Well, what does that mean? Linear independence means that if:

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

for all input values x then the coefficients above must all be 0. We can check to see if a set of functions is linearly independent by calculating the Wronskian for those functions. The Wronskian is defined as:

$$W(y_1, y_2, \ldots, y_n) = det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

where we note that the Wronskian is itself a function. If the Wronskian is *ever* equal to 0 on our interval of interest then it is equal to 0 on that entire interval and the functions are not linearly independent. Conversely, if the Wronskian is not 0 anywhere on the interval then it's not 0 everywhere on the interval and the functions $\{y_1, y_2, \ldots, y_n\}$ are linearly independent.

Note that for a second-order linear ODE we don't have to go through the whole Wronskian business. Two functions are linearly independent if one is not a constant multiple of the other.

5.3 Homogenous Equations with Constant Coefficients

When the functions $a_i(x)$ in our linear homogenous ODE are constants we can actually always solve the ODE. The basic idea is that if we have a differential equations in the form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots a_1y' + a_0y = 0$$

we get the associated "characteristic equation":

$$r^n + a_{n-1}r^{n-1} + \cdots a_1r + a_0 = 0.$$

Now, what we need to do is to find the roots of this equation. The solution to our ODE will just be exponentials with coefficients in the exponentials equal to the roots of our characteristic equation. If we have a repeated root, we need to take the solution we'd get with a single root and multiply it by x, x^2, etc... until we have a number of independent solutions equal to the order of the repeated root. If we have a complex root we'll always have its complex conjugate as a root as well, and the two complex conjugate roots will give us (possibly) a real exponential term multiplied by a sine function and a cosine function. This is difficult to understand when stated abstractly, but it's not that hard to do in practice. If you do 10 or so of them you should be in excellent shape. Here's an example of one:

What are the solutions to the differential equation:

$$y^{(5)} + 3y^{(4)} + 2y^{(3)} + 6y'' + y' + 3y = 0$$

the associated characteristic equation is:

$$r^5 + 3r^4 + 2r^3 + 6r^2 + r + 3 = 0$$

which factors as $(r^2+1)^2(r+3) = 0$ and so it has roots $\{-3, i, -i\}$, where the two complex roots are repeated. So, the homogenous solutions will be of the form:

$$y = c_1 e^{-3x} + c_2 \cos x + c_3 \sin x + c_4 x \cos x + c_5 x \sin x.$$

A more detailed explanation of the method, along with many more examples and practice problems, can be found in section 3.3 of the text.

5.4 Nonhomogenous Linear ODEs with Constant Coefficients

We learned two major techniques for dealing with nonhomogenous linear ODEs with constant coefficients. The first is the method of undertermined coefficients, which works if our function $f(x)$ is of some particular forms. The other is the method of variation of paramters, which works whenever we can solve the corresponding homogenous equation, but can be kind of difficult.

5.4.1 Method of Undetermined Coefficients

This method works when the function $f(x)$ in the equation:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots a_1 y' + a_0 y = f(x)$$

is either a polynomial, an exponential, or a trigonometric function. If it's a polynomial, you just guess that the particular solution is a polynomial of the same order. If it's an exponential, you just guess the particular solution is an exponential of the same form. If it's a cosine or sine, you just guess the particular solution is the sum of cosines and sines. If the function $f(x)$ is a combination, either multiplied or added, of solutions of this form, you just guess the particular solution is a combination of the same form. The one caveat is that if one of your guesses is not linearly independent of your homogenous solution, you need to keep multiplying it by x until it is. This is a whirlwind tour through the method, so I'd suggest reading through it again in section 3.5 if this is confusing for you. It's really not that hard, you just need to do some problems to get the hang of it.

5.4.2 Method of Variation of Paramters

This is a method that works in general. However, it's kind of technically complicated, so we only covered the method for second order linear ODEs. Say you've got a second order ODE, and you know the homogenous solution:

$$y_h = c_1 y_1 + c_2 y_2$$

then you guess that the particular solution will be of the form:

$$y_p = u_1 y_1 + u_2 y_2$$

and figure out what the functions u_1 and u_2 are. We derived in class, and it's derived in the textbook, that it works out to be:

$$y_p(x) = y_1(x) \int \frac{y_2(x) f(x)}{W(y_1, y_2)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(y_1, y_2)} dx$$

As mentioned, both of these methods are covered in section 3.5 of the textbook, which I'd recommend reading through once again before the test. I'd also recommend working some of the recommended practice problems.

6 Applications

A few of the sections in the textbook covered some major applications of differential equations, especially to population models and mechanical systems.

6.1 Population Models

There are two major population models that we discussed. The first is the exponential growth model:

$$\frac{dP}{dt} = kP$$

which predicts an exponentially increasing population. The other, more realistic population model, is the logistic population model:

$$\frac{dP}{dt} = kP(M - P)$$

which has a solution that starts out growing quickly (if the initial conditions is between 0 and M) but then slows down and asymptotically approaches M as time goes on. Note that this equation is in the form of the equations we used for studying equilibrium solutions and stability, and so those methods and ideas can be applied to the study of population models. More on this can be found in section 2.1 of the textbook.

6.2 Mechanical Models

The basic equations for all our mechanical models is Newton's second law, which states that the sum of the forces acting on an object is equal to the object's mass multiplied by its acceleration. Now, acceleration is the derivative of velocity, and velocity is the derivative of position, so frequently when modeling mechanical systems we end up with first or second order differential equations.

Here are some examples:

Assume that a body moving with velocity v encounters resistance of the form $\frac{dv}{dt} = -kv^{\frac{3}{2}}$. Show that

$$v(t) = \frac{4v_0}{(kt\sqrt{v_0} + 2)^2}.$$

Well, we're told that $\frac{dv}{dt} = -kv^{\frac{3}{2}}$, and so we get a separable differential equation:

$$\frac{dv}{v^{\frac{3}{2}}} = -kdt$$

which if we integrate both sides we get:

$$\frac{-2}{\sqrt{v}} = -kt + C$$

Now, if we solve this for v we get:

$$v(t) = \frac{4}{(kt + C)^2}$$

and if $v(0) = v_0$ we get $v_0 = \dfrac{4}{C^2}$ and so $C = \dfrac{2}{\sqrt{v_0}}$. If we plug this in we get:

$$v(t) = \frac{4}{(kt + \frac{2}{\sqrt{v_0}})^2} = \frac{4v_0}{(kt\sqrt{v_0} + 2)^2}$$

which is what we wanted to prove. More problems of this type and explanation of mechanical systems can be found in section 2.3.

Now, the mechanical system with which we dealt most closely was the mass-spring-dashpot system pictured below. If we treat the equilitribum position as $x = 0$ then an analysis of these forces yields the second-order linear homogenous differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

which has three types of solutions, depending on the roots of the characteristic equation, which are determined by the coefficients m, c, k. The three possitiblities are called overdamped, critically damped, and underdamped. When we have an underdamed system, we get oscillations. Much more explanation of this system can be found in section 3.4 of the text.