Math 2280 - Final Exam Part 1

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Name: Solutions

50 Points Possible

Note - This is the half of the exam that is closed book. Before you open your book you need to hand this part of the exam in. You can look at the second half and start working on it before you hand in this part, but you can't open your book until you hand this part in.

1. Answer the following:

a) What is the order of the following differential equation: (4 points)

$$2 \sin x y^{(3)} - y'y'' + x^5 y - 2e^{\cos x^2} = 15y'$$

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b) Circle the terms that describe the following differential equation: (2 points)

$$\sin x y^3 + 4e^x y = 15x^2$$

linear / nonlinear

homogeneous / nonhomogeneous

2. Solve the following initial value problem: (6 points)

$$y' + 5y = 0$$
$$y(0) = 7$$

$$\frac{dy}{dx} + 5y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -5y$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int -5\,dx$$

$$\Rightarrow \quad \ln|y| = -5x + C$$

$$\Rightarrow \quad y = Ce^{-5x}$$

$$y(0) = 7 = C.$$

So, $\boxed{y(x) = 7e^{-5x}}$

3. Using Euler's method with a step size of 1 estimate the value of $y(2)$ given the initial value problem: (6 points)

$$\frac{dy}{dx} = x^2 + y^2 - 2xy$$

$$y(0) = 2$$

$$\frac{dy}{dx}(0,2) = 0^2 + 2^2 - 2(0)(2) = 4$$

$$y(1) \approx 2 + 4 \cdot (1) = 6$$

$$\frac{dy}{dx}(1,6) = 1^2 + 6^2 - 2(1)(6) = 25$$

$$y(2) \approx 6 + 1 \times 25 = \boxed{31}$$

4. Solve the following initial value problem: (7 points)

$$y'' + 3y' - 10y = 0$$

$$y(0) = 2 \; y'(0) = 1$$

The characteristic equation is:

$$r^2 + 3r - 10 = 0$$

$$\Rightarrow (r+5)(r-2) = 0$$

So, we have roots $r = \{-5, 2\}$

and our general solution is:

$$y(x) = C_1 e^{-5x} + C_2 e^{2x}$$

$$y'(x) = -5C_1 e^{-5x} + 2C_2 e^{2x}$$

$$y(0) = C_1 + C_2 = 2$$

$$y'(0) = -5C_1 + 2C_2 = 1$$

$$\Rightarrow -5C_1 + 2(2 - C_1) = 1$$

$$\Rightarrow 4 - 7C_1 = 1 \Rightarrow C_1 = \frac{3}{7}$$

$$C_2 = 2 - C_1 = \frac{11}{7}$$

So,

$$\boxed{y(x) = \frac{3}{7} e^{-5x} + \frac{11}{7} e^{2x}}$$

5. Solve the following system of differential equations with the given initial values: (7 points)

$$x_1' = x_1 + 8x_2$$
$$x_2' = 2x_1 + x_2$$
$$x_1(0) = 0 \; x_2(0) = 4$$

$$\vec{x}' = \begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix} \vec{x}$$

Finding the eigenvalues of the matrix:

$$\begin{vmatrix} 1-\lambda & 8 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16 = \lambda^2 - 2\lambda - 15$$
$$(\lambda - 5)(\lambda + 3) = 0$$

So, the eigenvalues are $\lambda = \{5, -3\}$

The corresponding eigen vectors are:

$\lambda = 5$

$$\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 5 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$\Rightarrow -4a_1 + 8a_2 = 0$ $\quad \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ works.

$2a_1 - 4a_2 = 0$

Continued...

$\lambda = -3$

$$\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = -3\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 4b_1 + 8b_2 &= 0 \\ 2b_1 + 4b_2 &= 0 \end{aligned} \qquad \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ works.}$$

So,

$$\vec{x}(t) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}$$

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2c_1 - 2c_2 \\ c_1 + c_2 \end{pmatrix}$$

So, $c_1 = c_2 = 2$.

$$\boxed{\vec{x}(t) = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}}$$

6. Using the definition of the Laplace transform calculate the Laplace transform of the following function: (6 points)

$$f(t) = 2t$$

$$\mathcal{L}(2t) = \int_0^\infty 2t\, e^{-st} dt = 2\int_0^\infty t e^{-st} dt$$

$$u = t \quad dv = e^{-st} dt$$

$$du = dt \quad v = -\frac{e^{-st}}{s}$$

$$\Rightarrow 2\int_0^\infty t e^{-st} dt = -\frac{2te^{-st}}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st} dt$$

$$= 0 + \frac{2}{s}\left(-\frac{e^{-st}}{s}\right)\Big|_0^\infty$$

$$= \boxed{\frac{2}{s^2}}$$

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7. Determine if the point $x = 0$ in the following second order differential equation is an ordinary point, a regular singular point, or an irregular singular point. (6 points)

$$x^3y'' + 6\sin xy' + 6xy = 0$$

Rewriting:

$$y'' + \frac{6\sin x}{x^3}y' + \frac{6}{x^2}y = 0$$

$\lim_{x \to 0} \frac{6}{x^2}$ is undefined. So, $x = 0$ is a singular point.

Now,

$$p(x) = x\left(\frac{6\sin x}{x^3}\right) = \frac{6\sin x}{x^2}$$

$$\lim_{x \to 0} \frac{6\sin x}{x^2} = \lim_{x \to 0} \frac{6\cos x}{2x} = \frac{3}{0} \text{ undefined}$$

L'Hospital's Rule

So, $x = 0$ is an $\boxed{\text{irregular singular point}}$

8. Graph the even extension of the following function:

$$f(t) = t$$
$$0 \leq t \leq 1$$