Math 2280 - Final Exam Part 2

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Name: Solutions

50 Points Possible

Note - This is the half of the exam that is open book. You may look at this half of the exam, and even start it if you'd like, before you hand in the first half. However, before you open the book to work on this half you must hand in the first half.

1. Find the general solution to the following ODE: (10 points)

$$2xy^2 + (2x^2y + 4y^3)y' = -3x^2$$

$$2xy^2 + (2x^2y + 4y^3)\left(\frac{dy}{dx}\right) = -3x^2$$

$$\Rightarrow (2xy^2+3x^2)dx + (2x^2y+4y^3)dy = 0$$

$$M(x,y) = 2xy^2+3x^2$$
$$N(x,y) = 2x^2y + 4y^3$$

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}$$

So, the equation is exact.

$$\int M(x,y)dx = x^2y^2+x^3+C(y) = F(x,y)$$

$$\frac{\partial F(x,y)}{\partial y} = 2x^2y + C'(y) = 2x^2y+4y^3$$

$$\Rightarrow C'(y) = 4y^3$$

$$\Rightarrow C(y) = y^4 + C.$$

So, $\boxed{F(x,y) = x^2y^2+x^3+y^4 = C}$

implicitly solves the differential equation.

2. Find the general solution to the following ODE: (10 points)

$$y'' + 9y = 2x^2e^{3x} + 5$$

The homogenous solution is:

$$y_h(x) = c_1 \sin(3x) + c_2 \cos(3x).$$

For the particular solution we guess it's of the form:

$$y_p(x) = (Ax^2 + Bx + C)e^{3x} + D$$

$$y_p'(x) = 3(Ax^2 + Bx + C)e^{3x} + (2Ax + B)e^{3x}$$

$$y_p''(x) = 9(Ax^2 + Bx + C)e^{3x} + 3(2Ax + B)e^{3x} + 2Ae^{3x} + 3(2Ax + B)e^{3x}$$

Grouping terms:

$$y_p(x) = (Ax^2 + Bx + C)e^{3x} + D$$

$$y_p'(x) = [3Ax^2 + (2A + 3B)x + (B + 3C)]e^{3x}$$

$$y_p''(x) = [9Ax^2 + (12A + 9B)x + (2A + 6B + 9C)]e^{3x}$$

Plugging these into the ODE we get:

$$[18Ax^2 + (12A + 18B)x + (2A + 6B + 18C)]e^{3x} + 9D$$

So,

$$18A = 2 \quad \Rightarrow \quad A = \frac{1}{9}$$

$$12A + 18B = 0 \qquad B = -\frac{2}{27}$$

$$2A + 6B + 18C = 0 \qquad C = \frac{1}{81}$$

$$9D = 5 \qquad D = \frac{5}{9}$$

Continued...

So,

$$y(x) = y_h(x) + y_p(x)$$

$$= \boxed{C_1 \sin(3x) + C_2 \cos(3x) + \left(\frac{1}{9}x^2 - \frac{2}{27}x + \frac{1}{81}\right)e^{3x} + \frac{5}{9}}$$

3. Find the general solution to the system of ODEs described by the following matrix equation: (10 points)

$$\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \mathbf{x}$$

Finding the eigenvalues:

$$\begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 4$$

$$= 21 - 10\lambda + \lambda^2 + 4$$

$$= \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

So, the matrix has a repeated eigenvalue $\lambda = 5$.

Solving for the eigenvector:

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 5 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 2a_1 + a_2 = 0 \\ -4a_1 - 2a_2 = 0 \end{array} \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ works}$$

We cannot find another linearly independent eigenvector, so we must find a vector $\vec{v}_2$ such that

$$(A - 5I)^2 \vec{v}_2 = 0$$

Continued...

and

$$(A-5I)\vec{v}_2 = \vec{v}_1 .$$

Now, $(A-5I)^2 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

So, any $\vec{v}_2$ will work as long as $\vec{v}_2 \neq 0$, and $(A-5I)\vec{v}_2 \neq 0$.

Take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ which is twice the $\vec{v}_1$ we calculated earlier.

So, we get the solutions:

$$\vec{x}_1(t) = \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{5t}$$

$$\vec{x}_2(t) = \left(\begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^{5t} .$$

So, our general solution is:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{5t} + c_2\left[\begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] e^{5t}}$$

4. Use Laplace transforms to solve the ODE: (10 points)

$$x'' + 2x + x = \delta(t) - \delta(t-2)$$
$$x(0) = x'(0) = 2$$

Taking the Laplace transform of both sides we get:

$$\mathcal{L}(x'') = s^2 X(s) - 2s - 2 \qquad \mathcal{L}(\delta(t)) = 1$$
$$\mathcal{L}(x') = s\,X(s) - 2 \qquad \mathcal{L}(\delta(t-2)) = e^{-2s}$$
$$\mathcal{L}(x) = X(s)$$

So, we get:

$$s^2 X(s) - 2s - 2 + 2sX(s) - 4 + X(s) = 1 - e^{-2s}$$

$$\Rightarrow (s^2+2s+1)X(s) = 2s + 7 - e^{-2s}$$

$$\Rightarrow X(s) = \frac{2s}{(s+1)^2} + \frac{7}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}$$

$$\frac{2s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2} = \frac{As + (A+B)}{(s+1)^2} \Rightarrow \begin{matrix} A = 2 \\ B = -2 \end{matrix}$$

$$= \frac{2}{s+1} - \frac{2}{(s+1)^2}$$

So,

$$X(s) = \frac{2}{s+1} + \frac{5}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}$$

Continued..

The inverse Laplace transforms of the first two terms are:

$$\mathcal{L}^{-1}\left(\frac{2}{s+1}\right) = 2e^{-t}$$

$$\mathcal{L}^{-1}\left(\frac{5}{(s+1)^2}\right) = 5te^{-t}$$

where as:

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s+1)^2}\right) = u(t-2)(t-2)\,e^{-(t-2)}$$

So,

$$\boxed{x(t) = 2e^{-t} + 5te^{-t} - u(t-2)(t-2)\,e^{-(t-2)}}$$

5. Use either power series or Frobenius series methods to construct two linearly independent solutions to the differential equation: (10 points)

$$2xy'' + (1 - 2x^2)y' - 4xy = 0$$

Rewriting:

$$y'' + \left(\frac{1-2x^2}{2x}\right)y' - 2y = 0$$

$\lim_{x \to 0} \left(\frac{1-2x^2}{2x}\right)$ is undefined, so $x = 0$ is a singular point.

$$p(x) = x\left(\frac{1-2x^2}{2x}\right) = \frac{1}{2} - x^2 \qquad \lim_{x \to 0} p(x) = \frac{1}{2}$$

$$q(x) = x^2(-2) = -2x^2 \qquad \lim_{x \to 0} q(x) = 0.$$

So, it's a regular singular point with indicial equation:

$$r(r-1) + \frac{1}{2}r = 0 \Rightarrow r\left(r - \frac{1}{2}\right) = 0.$$

So, $r = \{0, \frac{1}{2}\}$ are the exponents

Now,

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

Continued...

Plugging these into the ODE we get:

$$\sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$$
$$-\sum_{n=0}^{\infty} 2c_n(n+r)x^{n+r+1} - \sum_{n=0}^{\infty} 4c_n x^{n+r+1}$$

The x^{r-1} term is handled by the indicial equation, so c_0 is "arbitrary" for both values of r.

The x^r term gives the relation:

$$2c_1(1+r)r + c_1(1+r) = 0 \Rightarrow (2r+1)(r+1)c_1 = 0.$$

For $r=0$ or $r=\frac{1}{2}$ we must have $c_1 = 0$.

For the higher order terms we get the recurrence relation:

$$2c_{n+2}(n+r+2)(n+r+1) + c_{n+2}(n+r+2) - 2c_n(n+r) - 4c_n = 0.$$
$$\Rightarrow c_{n+2}(n+r+2)(2n+2r+3) - (2n+2r+4)c_n = 0.$$
$$\Rightarrow c_{n+2} = \frac{2c_n}{2n+2r+3}$$

For $r=0$ we get:

$c_0 = c_0$

$c_2 = \frac{2c_0}{3}$

$c_4 = \frac{2c_2}{7} = \frac{2^2 c_0}{7\cdot 3}$

$c_6 = \frac{2c_4}{11} = \frac{2^3 c_0}{11\cdot 7\cdot 3}$

In general $n \geq 1$

$c_{2n} = \frac{2^n c_0}{1\cdot 3\cdot 7\cdot 11 \cdots (4n-1)}$

For $r=\frac{1}{2}$ we get

$c_0 = c_0$

$c_2 = \frac{c_0}{2}$

$c_4 = \frac{c_2}{4} = \frac{c_0}{4\cdot 2}$

$c_6 = \frac{c_4}{6} = \frac{c_0}{6\cdot 4\cdot 2}$

In general $n \geq 0$

$c_{2n} = \frac{c_0}{2^n n!}$

So,

$$\boxed{y(x) = c_1\left(1 + \sum_{n=1}^{\infty} \frac{2^n x^{2n}}{3\cdot 7\cdot 11 \cdots (4n-1)}\right) + c_2\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}\right) = c_1\left(1 + \sum_{n=1}^{\infty} \frac{2^n x^{2n}}{3\cdot 7\cdot 11 \cdots (4n-1)}\right) + c_2 e^{\frac{x^2}{2}}}$$

6. *Extra Credit* In this problem we will step through a derivation of a closed form equation for the nth term of the Fibonacci sequence. (10 points)

   a) The Fibonacci sequence is defined by the recurrence relation:

   $$x_{n+2} = x_{n+1} + x_n$$

   with $x_0 = 0$ and $x_1 = 1$.

   Suppose that we have a solution in the form $x_n = r^n$. If this "guess" is correct what must the value of r be for it to work? (There will be 2 possible values). (3 points)

$$r^{n+2} = r^{n+1} + r^n$$

$$\Rightarrow r^2 = r + 1 \Rightarrow r^2 - r - 1 = 0.$$

So, $$r = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

So, if $r = \frac{1 \pm \sqrt{5}}{2}$ then it satisfies

$$r^{n+2} = r^{n+1} + r^n.$$

b) In fact, any linear combination of the above two possible values of r will also satisfy the Fibonacci relation in the form $x_n = Ar_1^n + Br_2^n$. Find out what linear combination we need to use in order for it to square with our initial conditions, and from this derive a closed form solution for x_n. (4 points)

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

$$A r_1^0 + B r_2^0 = 0$$

$$A r_1 + B r_2 = 1$$

$$\Rightarrow \quad A + B = 0$$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\Rightarrow \quad A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right) = 1 \Rightarrow A = \frac{1}{\sqrt{5}}, \quad B = -\frac{1}{\sqrt{5}}.$$

So,

$$\boxed{X_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

c) Define $y_n = \frac{x_{n+1}}{x_n}$. Figure out $\lim_{n\to\infty} y_n$. (2 points)

Note

$$\lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n = 0 \quad \text{as} \quad \left|\frac{1-\sqrt{5}}{2}\right| < 1.$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}} = \frac{1+\sqrt{5}}{2}.$$

d) What famous number is this? (1 point)

ϕ the Golden Ratio a.k.a. the Golden Mean.