

# Assignment #9 Solutions

## Section 5.5

5.5.1 Find a fundamental matrix and a solution to the system  
$$\vec{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{x} \quad \vec{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Solve for the eigenvalues of the matrix:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

$$\Rightarrow 2-\lambda = \pm 1 \Rightarrow \boxed{\lambda = 1, 3}$$

Finding an eigenvector for  $\lambda = 1$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{array}{l} a+b=0 \\ a+b=0 \end{array} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works}$$

Finding an eigenvector for  $\lambda = 3$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{array}{l} -a+b=0 \\ -a+b=0 \end{array} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

So, we get two linearly independent solutions:

$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$$\vec{x}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

And a fundamental matrix:

$$\Phi(t) = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix}$$

Now,  $\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

$$\Rightarrow \Phi(0)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \vec{c} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix}$$

and so,

$$\vec{x}(t) = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{5}{2}e^t + \frac{1}{2}e^{3t} \\ -\frac{5}{2}e^t + \frac{1}{2}e^{3t} \end{pmatrix}}$$

5.5.7.

$$\vec{x}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \vec{x} \quad \vec{x}(0) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

First, we find the eigenvalues of the matrix:

$$\begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix}$$

$$= (5-\lambda)[(1+\lambda)(4+\lambda)-4] - 6[-4+4(1+\lambda)]$$

$$= (5-\lambda)[\lambda^2+5\lambda] - 24\lambda$$

$$= 5\lambda^2+25\lambda - \lambda^3 - 5\lambda^2 - \cancel{24\lambda}$$

$$= \cancel{+9\lambda - \lambda^3} \lambda - \lambda^3$$

$$= \lambda(1-\lambda)(1+\lambda)$$

So, we have eigenvalues  $\lambda = 0, -1, 1$

For  $\lambda = 0$  we have the eigenvector

$$\begin{aligned} 5a - 6c &= 0 \\ 2a - b - 2c &= 0 \\ 4a - 2b - 4c &= 0 \end{aligned} \quad \vec{x} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \text{ works.}$$

For  $\lambda = 1$  we have:

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{aligned} 4a - 6c &= 0 \\ 2a - 2b - 2c &= 0 \\ 4a - 2b - 4c &= 0 \end{aligned} \quad \vec{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \text{ works.}$$

For  $\lambda = -1$  we have:

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = - \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{aligned} 6a - 6c &= 0 \\ 2a - 2c &= 0 \\ 4a - 2b - 3c &= 0 \end{aligned} \quad \vec{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ works}$$

So, a fundamental matrix is:

$$\Phi(t) = \begin{pmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{pmatrix}$$

To find a particular solution:

$$\Phi(0) = \begin{pmatrix} 6 & 3 & 2 \\ 2 & 1 & 1 \\ 5 & 2 & 2 \end{pmatrix}$$

using Matlab

$$I(0)^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

and

$$x(t) = \begin{pmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{pmatrix} \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} -12 + 12e^t + 2e^{-t} \\ -4 + 4e^t + e^{-t} \\ -10 + 8e^t + 2e^{-t} \end{pmatrix}$$

5.5.9

Compute the matrix exponential  $e^{At}$  for each system  $\vec{x}' = A\vec{x}$ .

$$\begin{aligned}x_1' &= 5x_1 - 4x_2 \\x_2' &= 2x_1 - x_2\end{aligned}$$

$$\vec{A} = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

Finding the eigenvalues:

$$\begin{aligned}\begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} &= (\lambda-5)(\lambda+1) + 8 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda-3)(\lambda-1)\end{aligned}$$

$$\lambda = 3, 1$$

Finding two eigenvectors for our eigenvalues:

For  $\lambda=3$

$$\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{aligned}2a - 4b &= 0 \\ 2a - 4b &= 0\end{aligned} \quad \vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ works.}$$

For  $\lambda=1$

$$\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{aligned}4a - 4b &= 0 \\ 2a - 2b &= 0\end{aligned} \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

So, a fundamental matrix is:

$$\Phi(t) = \begin{pmatrix} 2e^{3t} & e^t \\ e^{3t} & e^t \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Phi^{-1}(0) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

and so

$$e^{At} = \Phi(t) \Phi(0)^{-1} = \begin{pmatrix} 2e^{3t} & e^t \\ e^{3t} & e^t \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{3t} - e^t & -2e^{3t} + 2e^t \\ e^{3t} - e^t & -e^{3t} + 2e^t \end{pmatrix}$$

5.5.18

$$\begin{aligned}x_1' &= 4x_1 + 2x_2 \\x_2' &= 2x_1 + 4x_2\end{aligned}$$

$$\vec{x}' = A\vec{x} \quad A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

Finding the eigenvalues:

$$\begin{aligned}\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} &= (4-\lambda)^2 - 4 \\ &= \lambda^2 - 8\lambda + 12 \\ &= (\lambda - 6)(\lambda - 2)\end{aligned}$$

So, the eigenvalues are  $\lambda = 2$  and  $\lambda = 6$ .

Finding the eigenvectors:

$$\lambda = 2$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{aligned}2a + 2b &= 0 \\ 2a + 2b &= 0\end{aligned} \quad \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works.}$$

$$\lambda = 6$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 6 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\rightarrow \begin{aligned}-2a + 2b &= 0 \\ 2a - 2b &= 0\end{aligned} \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

So, we have fundamental matrix:

$$\Phi(t) = \begin{pmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\Phi(0)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} e^{2t} + \frac{1}{2} e^{6t} & -\frac{1}{2} e^{2t} + \frac{1}{2} e^{6t} \\ -\frac{1}{2} e^{2t} + \frac{1}{2} e^{6t} & \frac{1}{2} e^{2t} + \frac{1}{2} e^{6t} \end{pmatrix}$$

5.5.24

Show that the matrix  $A$  is nilpotent and then use this fact to find the matrix exponential  $e^{At}$ .

$$A = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So,

$$e^{At} = I + At + \frac{1}{2!} A^2 t^2$$

$$= \begin{pmatrix} 1+3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{pmatrix} = \begin{pmatrix} 1-3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{pmatrix}$$

5.6.1

Use the method of undetermined coefficients to find a particular solution. If initial conditions are given, find a particular solution that satisfies these conditions.

$$\begin{aligned}x' &= x + 2y + 3 \\y' &= 2x + y - 2\end{aligned}$$

$$\Rightarrow \vec{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

First, we solve the homogenous equation:

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x}$$

Finding the eigenvalues

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow (1-\lambda)^2 = 4$$

$$\Rightarrow (1-\lambda) = \pm 2$$

$$\lambda = \{-1, 3\}$$

Finding eigen vectors

For  $\lambda = -1$ .

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{aligned}\Rightarrow 2a + 2b &= 0 \\ 2a + 2b &= 0\end{aligned}$$

$$\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works.}$$

For  $\lambda = 3$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{aligned} -2a + 2b &= 0 & \vec{x} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.} \\ 2a - 2b &= 0 \end{aligned}$$

So, our homogenous general solution is:

$$\vec{x}_h(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

Now, for our particular solution we note that it's a constant "polynomial", and so we "guess" our solution is too:

$$\vec{x}_p = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \vec{x}_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{aligned} c_1 + 2c_2 + 3 &= 0 \\ 2c_1 + c_2 - 2 &= 0 \end{aligned}$$

$$-3c_1 + 7 = 0 \Rightarrow c_1 = \frac{7}{3}$$

$$\Rightarrow \frac{6}{7} + c_2 - 2 = 0 \Rightarrow c_2 = \frac{8}{7}$$
$$\frac{14}{3} + c_2 - 2 = 0 \Rightarrow c_2 = -\frac{8}{3}$$

So,

$$\vec{x}_p = \frac{1}{3} \begin{pmatrix} 7 \\ -8 \end{pmatrix}$$

Thus, our solution is:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \frac{1}{3} \begin{pmatrix} 7 \\ -8 \end{pmatrix}$$

The particular solution can be expressed:

$$x(t) = \frac{7}{3} \quad y(t) = -\frac{8}{3}$$

5. 6. 6.

$$\begin{aligned} x' &= 9x + y + 2e^t \\ y' &= -8x - 2y + te^t \end{aligned}$$

So,

$$\vec{x}' = \begin{pmatrix} 9 & 1 \\ -8 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^t \\ te^t \end{pmatrix}$$

First we solve the homogenous problem

$$\vec{x}' = \begin{pmatrix} 9 & 1 \\ -8 & -2 \end{pmatrix} \vec{x}$$

Finding the eigenvalues:

$$\begin{aligned} \begin{vmatrix} 9-\lambda & 1 \\ -8 & -2-\lambda \end{vmatrix} &= (9-\lambda)(-2-\lambda) + 8 \\ &= -18 - 9\lambda + 2\lambda + \lambda^2 + 8 \\ &= \lambda^2 - 7\lambda - 10 \end{aligned}$$

$$\lambda = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(-10)}}{2} = \frac{7 \pm \sqrt{89}}{2}$$

We note that neither of these  $\lambda=1$ , so there is no interference with our particular solution.

We "guess" that the form of our particular solution is:

$$\vec{x}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} t e^t + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} e^t$$

$$\vec{x}_p' = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} t e^t + \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} e^t$$

So, we get the relations from our system:

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} t e^t + \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} e^t = \begin{pmatrix} 9 & 1 \\ -8 & -2 \end{pmatrix} \left[ \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} t e^t + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} e^t \right] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t e^{t/2} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, we end up with the 4 relations:

$$\begin{aligned} a_1 &= 9a_1 + b_1 \\ b_1 &= -8a_1 - 2b_1 + 1 \\ a_1 + a_2 &= 9a_2 + b_2 + 2 \\ b_1 + b_2 &= -8a_2 - 2b_2 \end{aligned}$$

Solving first for  $a_1$  and  $b_1$

$$b_1 = -8a_1$$

$$\Rightarrow b_1 = b_1 - 2b_1 + 1 \Rightarrow 2b_1 = 1 \Rightarrow b_1 = \frac{1}{2} \quad a_1 = -\frac{1}{16}$$

and using these values to solve for  $a_2, b_2$  we get:

$$a_2 = -\frac{91}{256} \quad b_2 = \frac{25}{32}$$

So,

$$\boxed{\vec{x}_p = \frac{1}{16} \begin{pmatrix} -1 \\ 8 \end{pmatrix} t e^t + \frac{1}{256} \begin{pmatrix} -91 \\ 200 \end{pmatrix} e^t}$$

5.6.10.

$$\begin{aligned}x' &= x - 2y \\y' &= 2x - y + e^t \sin(t)\end{aligned}$$

First, we investigate the eigenvalues of our homogenous equation:

$$\vec{x}' = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \vec{x}$$

$$\begin{aligned}\begin{vmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} &= (1-\lambda)(-1-\lambda) + 4 \\ &= -1 - \lambda + \lambda + \lambda^2 + 4 \\ &= \lambda^2 + 3\end{aligned}$$

So, we have eigenvalues  $\lambda = \pm\sqrt{3}i$ , which is not  $\lambda = 1 \pm i$ , so we have no interference with our "guess" and our homogenous solution.

Our system is:

$$\vec{x}' = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \sin(t).$$

So, our "guess" will be

$$\vec{x}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^t \cos(t) + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} e^t \sin(t)$$

$$\vec{x}_p' = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} e^t \cos(t) + \begin{pmatrix} -a_1 + a_2 \\ -b_1 + b_2 \end{pmatrix} e^t \sin(t).$$

Plugging this into our system of ODEs we get:

$$\begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} e^t \cos(t) + \begin{pmatrix} -a_1 + a_2 \\ -b_1 + b_2 \end{pmatrix} e^t \sin(t) \\ = \begin{pmatrix} a_1 - 2b_1 \\ 2a_1 - b_1 \end{pmatrix} e^t \cos(t) + \begin{pmatrix} a_2 - 2b_2 \\ 2a_2 - b_2 + 1 \end{pmatrix} e^t \sin(t)$$

So, we get the 4 equations:

$$\begin{aligned} a_1 + a_2 &= a_1 - 2b_1 & a_2 &= -2b_1 \\ b_1 + b_2 &= 2a_1 - b_1 & b_2 &= 2a_1 - 2b_1 \\ -a_1 + a_2 &= a_2 - 2b_2 & -a_1 &= -2b_2 \\ -b_1 + b_2 &= 2a_2 - b_2 + 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow -a_1 - 2b_1 &= -2b_1 - 2(2a_1 - 2b_1) \\ -b_1 + (2a_1 - 2b_1) &= 2(-2b_1) - (2a_1 - 2b_1) + 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow 3a_1 - 4b_1 &= 0 & \Rightarrow -12a_1 + 16b_1 &= 0 \\ 4a_1 - 3b_1 &= 1 & 12a_1 - 9b_1 &= 3 \end{aligned}$$

$$\begin{aligned} \Rightarrow 2b_1 &= 3 & \Rightarrow b_1 &= \frac{3}{2} \\ a_1 &= \frac{12}{21} = \frac{4}{7} \\ a_2 &= -\frac{6}{7} \\ b_2 &= \frac{2}{7} \end{aligned}$$

So,

$$\vec{x}_p(t) = \frac{1}{21} \begin{pmatrix} 12 \\ 9 \end{pmatrix} e^t \cos(t) + \frac{1}{7} \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^t \sin(t)$$

$$\begin{aligned} -a_1 - 2b_1 &= -2b_1 - 4a_1 + 4b_1 \\ 2a_1 - 3b_1 &= -4b_1 - 2a_1 + 2b_1 + 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow -a_1 - 2b_1 &= -4a_1 + 2b_1 \\ 2a_1 - 3b_1 &= -2a_1 - 2b_1 + 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow 3a_1 - 4b_1 &= 0 \\ 4a_1 - b_1 &= 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow 3a_1 - 4b_1 &= 0 \\ -16a_1 + 4b_1 &= -4 \end{aligned}$$

$$\Rightarrow -13a_1 = -4$$

$$\Rightarrow a_1 = 4/13$$

$$b_2 = 2/13$$

$$b_1 = a_1 - \frac{1}{2}b_2 = 3/13$$

$$a_2 = -6/13$$

So,

$$\boxed{x_p(t) = \frac{1}{13} \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^t \cos(t) + \frac{1}{13} \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^t \sin(t)}$$

i, 6.17

Use the method of variation of parameters to solve the initial value problem

$$\vec{x}' = A\vec{x} + \vec{f}(t), \quad \vec{x}(a) = \vec{x}_a.$$

$$A = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}, \quad \vec{f}(t) = \begin{pmatrix} 60 \\ 90 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$e^{At} = \frac{1}{6} \begin{pmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{pmatrix}$$

$$e^{-A(s-t)} = \frac{1}{6} \begin{pmatrix} -e^{(s-t)} + 7e^{-5(s-t)} & 7e^{s-t} - 7e^{-5(s-t)} \\ -e^{(s-t)} + e^{-5(s-t)} & 7e^{s-t} - e^{-5(s-t)} \end{pmatrix}$$

$$\vec{x}(t) = \int_0^t e^{-A(s-t)} \vec{f}(s) ds$$

$$= \int_0^t \begin{pmatrix} -e^{(s-t)} + 7e^{-5(s-t)} & 7e^{s-t} - 7e^{-5(s-t)} \\ -e^{(s-t)} + e^{-5(s-t)} & 7e^{s-t} - e^{-5(s-t)} \end{pmatrix} \begin{pmatrix} 10 \\ 15 \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} 95e^{(s-t)} - 35e^{-5(s-t)} \\ 95e^{(s-t)} - 5e^{-5(s-t)} \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} 95e^{-t}e^s - 35e^{5t}e^{-5s} \\ 95e^{-t}e^s - 5e^{5t}e^{-5s} \end{pmatrix} ds$$

$$= \begin{pmatrix} 95 + 7 \\ 95 + 1 \end{pmatrix} - \begin{pmatrix} 95e^{-t} + 7e^{5t} \\ 95e^{-t} + e^{5t} \end{pmatrix}$$

$$= \begin{pmatrix} 102 - 95e^{-t} - 7e^{5t} \\ 96 - 95e^{-t} - e^{5t} \end{pmatrix}$$

5.6.19

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \vec{f}(t) = \begin{pmatrix} 180t \\ 90 \end{pmatrix}, \vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$e^{At} = \frac{1}{5} \begin{pmatrix} e^{-3t} + 4e^{2t} & -2e^{-3t} + 2e^{2t} \\ -2e^{-3t} + 2e^{2t} & 4e^{-3t} + e^{2t} \end{pmatrix}$$

$$e^{-A(t-s)} = \frac{1}{5} \begin{pmatrix} e^{3(s-t)} + 4e^{-2(s-t)} & -2e^{3(s-t)} + 2e^{-2(s-t)} \\ -2e^{3(s-t)} + 2e^{-2(s-t)} & 4e^{3(s-t)} + e^{-2(s-t)} \end{pmatrix}$$

$$\vec{x}(t) = \int_0^t \begin{pmatrix} e^{3(s-t)} + 4e^{-2(s-t)} & -2e^{3(s-t)} + 2e^{-2(s-t)} \\ -2e^{3(s-t)} + 2e^{-2(s-t)} & 4e^{3(s-t)} + e^{-2(s-t)} \end{pmatrix} \begin{pmatrix} 36s \\ 18 \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} e^{-3t} (32s - 36) e^{3s} + e^{2t} (128s + 36) e^{-2s} \\ e^{-3t} (-64s + 72) e^{3s} + e^{2t} (64s + 18) e^{-2s} \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} e^{-3t} (36s - 36) e^{3s} + e^{2t} (144s + 36) e^{-2s} \\ e^{-3t} (-64s + 72) e^{3s} + e^{2t} (64s + 18) e^{-2s} \end{pmatrix} ds$$

$$= \left( \begin{pmatrix} e^{-3t} (12se^{3s} - 16e^{3s}) + e^{2t} (-72se^{-2s} - 54e^{-2s}) \\ e^{-3t} (-24se^{3s} + 32e^{3s}) + e^{2t} (-36se^{-2s} - 27e^{-2s}) \end{pmatrix} \right) \Big|_0^t$$

$$= \begin{pmatrix} 12t - 16 - 72t - 54 + 16e^{-3t} + 54e^{2t} \\ 24t + 32 - 36t - 27 + 32e^{-3t} + 27e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -60t - 70 + 16e^{-3t} + 54e^{2t} \\ -60t + 5 + 32e^{-3t} + 27e^{2t} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -60t - 70 + 16e^{-3t} + 54e^{2t} \\ -60t + 5 + 32e^{-3t} + 27e^{2t} \end{pmatrix}}$$