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Solution Set #8

5.2.1 Solve the system of ODEs and graph the corresponding direction field and typical solution curves.

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 + x_2$$

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$
$$= (\lambda - 3)(\lambda + 1)$$

So, we have eigenvalues $\lambda = 3, -1$ with corresponding eigenvectors

$\lambda = 3$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{matrix} a + 2b = 3a \\ 2a + b = 3b \end{matrix} \Rightarrow \begin{matrix} -2a + 2b = 0 \\ 2a - 2b = 0 \end{matrix}$$

So,

$a = b$ and an eigenvector

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} 2a + 2b = 0 \\ 2a + 2b = 0 \end{matrix} \quad \Rightarrow \quad a = -b$$

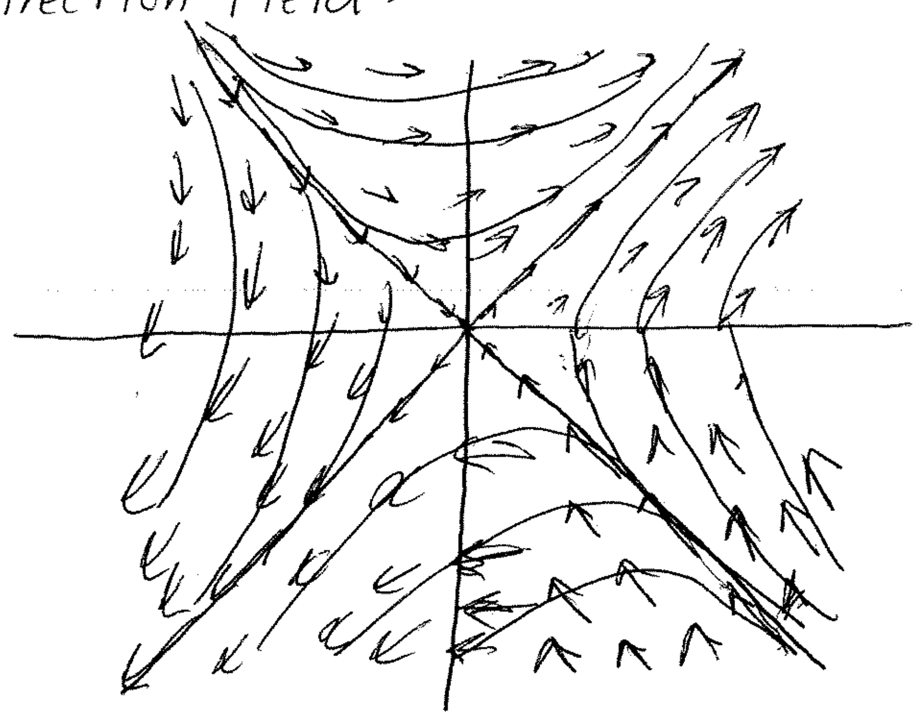
So,

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a working eigenvector.

So, we have general solution

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Direction Field:



5.2.9

$$x_1' = 2x_1 - 5x_2 \qquad x_1(0) = 2$$
$$x_2' = 4x_1 - 2x_2 \qquad x_2(0) = 3$$

$$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = \cancel{(2-\lambda)^2 + 20 = -\lambda^2 + 4\lambda + 16}$$
$$= (2-\lambda)(-2-\lambda) + 20$$
$$= -4 + \lambda^2 + 20$$

$$\Rightarrow \lambda^2 + 16 \Rightarrow \boxed{\lambda = \pm 4i}$$

So, we have complex eigenvalues.

$$\begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix} = 4i \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}$$

$$\Rightarrow (2a_1 - 5a_2) + i(2b_1 - 5b_2) = -4b_1 + i\,4a_1$$
$$(4a_1 - 2a_2) + i(4b_1 - 2b_2) = -4b_2 + i\,4a_2$$

So, we get the relations

$$2a_1 - 5a_2 = -4b_1$$
$$4a_1 - 2a_2 = -4b_2$$
$$2b_1 - 5b_2 = 4a_1$$
$$4b_1 - 2b_2 = 4a_2$$

One set that works is

$$a_1 = 5 \quad b_1 = 5$$
$$a_2 = 6 \quad b_2 = -2$$

$$\cancel{2b_1 - 5b_2 - 2b_1 + b_2 = -4b_2}$$

$\cancel{\Rightarrow}$

So, we get the general solution:

$$\boxed{\vec{x}(t) = c_1\left[\begin{pmatrix}5\\6\end{pmatrix}\cos(4t) - \begin{pmatrix}5\\-2\end{pmatrix}\sin(4t)\right] + c_2\left[\begin{pmatrix}5\\-2\end{pmatrix}\cos(4t) + \begin{pmatrix}5\\6\end{pmatrix}\sin(4t)\right]}$$

$x_1(0) = 2 \quad x_2(0) = 3$

$$\Rightarrow \begin{matrix} 5c_1 + 5c_2 = 2 \\ 6c_1 - 2c_2 = 3 \end{matrix} \Rightarrow \begin{matrix} 30c_1 + 30c_2 = 12 \\ 30c_1 - 10c_2 = 15 \end{matrix}$$

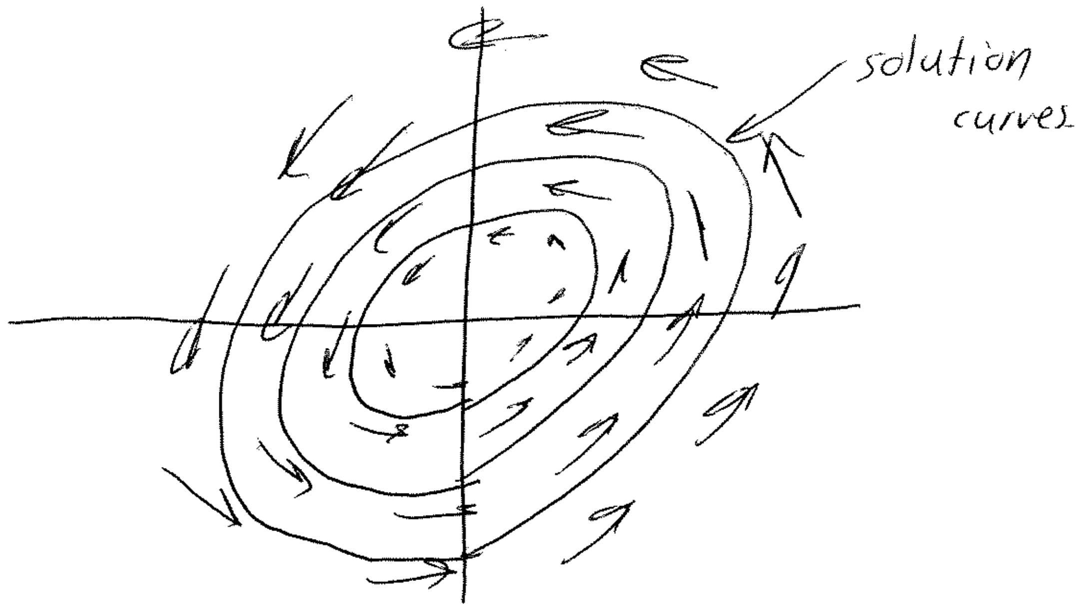
$$\Rightarrow 40c_2 = -3 \qquad c_2 = -3/40$$

$$c_1 = \frac{1}{5}\left(2 + \frac{3}{8}\right) = 19/40$$

So,

$$\vec{x}(t) = \frac{19}{40}\left[\begin{pmatrix}5\\6\end{pmatrix}\cos(4t) - \begin{pmatrix}5\\-2\end{pmatrix}\sin(4t)\right] - \frac{3}{40}\left[\begin{pmatrix}5\\-2\end{pmatrix}\cos(4t) + \begin{pmatrix}5\\6\end{pmatrix}\sin(4t)\right]$$

$$\Rightarrow \boxed{\vec{x}(t) = \begin{pmatrix} 2\cos(4t) - \frac{11}{4}\sin(4t) \\ 3\cos(4t) + \frac{1}{2}\sin(4t) \end{pmatrix}}$$



5.2.15 $x_1' = 7x_1 - 5x_2$

$x_2' = 4x_1 + 3x_2$

$$\vec{x}' = \begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix} \vec{x}$$

Solving for the eigenvalues:

$$\begin{vmatrix} 7-\lambda & -5 \\ 4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 20$$

$$= 21 - 10\lambda + \lambda^2 + 20$$

$$\Rightarrow \lambda^2 - 10\lambda + 41$$

$$\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(41)}}{2} = 5 \pm 4i$$

So, we want to find an eigenvector:

$$\begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix} = 5+4i \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} (7a_1 - 5a_2) + i(7b_1 - 5b_2) = (5a_1 - 4b_1) + i(4a_1 + 5b_1) \\ (4a_1 + 3a_2) + i(4b_1 + 3b_2) = (5a_2 - 4b_2) + i(4a_2 + 5b_2) \end{matrix}$$

Interestingly enough we see, again, that the set: $\begin{matrix} a_1 = 5 & b_1 = 5 \\ a_2 = 6 & b_2 = -2 \end{matrix}$

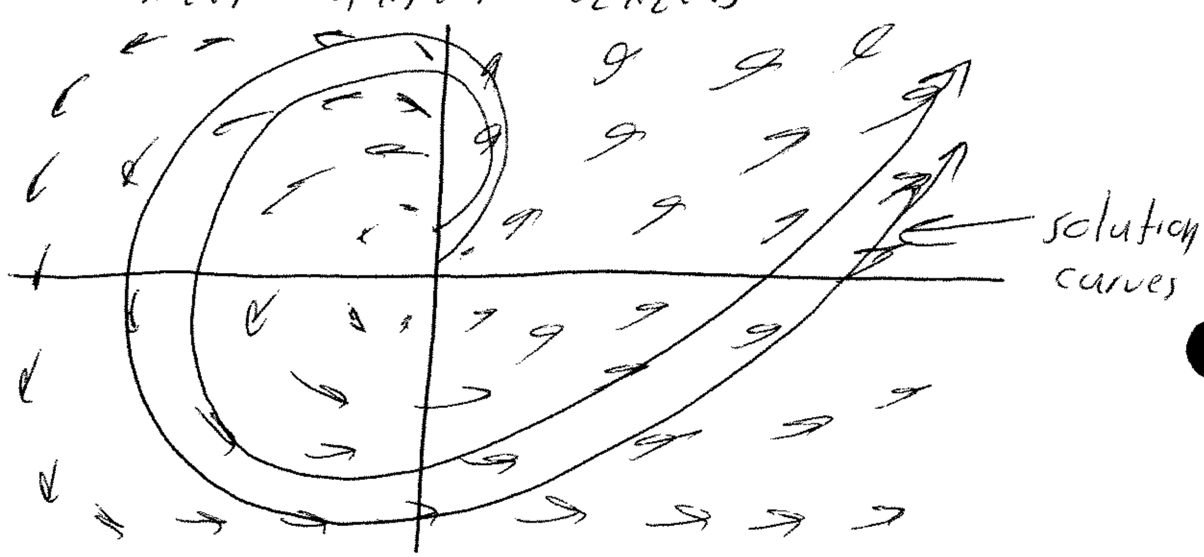
So, we have the solutions:

$$\vec{x}_1(t) = e^{5t}\left[\begin{pmatrix} 5 \\ 6 \end{pmatrix}\cos(4t) - \begin{pmatrix} 5 \\ -2 \end{pmatrix}\sin(4t)\right]$$

$$\vec{x}_2(t) = e^{5t}\left[\begin{pmatrix} 5 \\ -2 \end{pmatrix}\cos(4t) + \begin{pmatrix} 5 \\ 6 \end{pmatrix}\sin(4t)\right]$$

and the general solution:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$



5.2.21 Solve the system of equations:

$$\begin{aligned} x_1' &= 5x_1 \qquad\quad - 6x_3 \\ x_2' &= 2x_1 - x_2 - 2x_3 \\ x_3' &= 4x_1 - 2x_2 - 4x_3 \end{aligned}$$

$$\vec{x}\,' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix} = (5-\lambda)\left[(-1-\lambda)(-4-\lambda)-4\right] - 6\left[-4-(-1-\lambda)4\right]$$

$$\begin{aligned} &= (5-\lambda)(5\lambda+\lambda^2)+24+24(-1-\lambda) \\ &= 25\lambda+5\lambda^2-5\lambda^2-\lambda^3-24\lambda \\ &= -\lambda^3+\lambda = -\lambda(\lambda+1)(\lambda-1) \end{aligned}$$

So, we have eigenvalues $\lambda = 0, -1, 1$.

With associated eigenvectors:

For $\lambda = 0$:

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 5a \qquad - 6c &= 0 \\ 2a - b - 2c &= 0 \\ 4a - 2b - 4c &= 0 \end{aligned} \qquad \begin{aligned} a &= 6 \\ b &= 2 \\ c &= 5 \end{aligned} \quad \text{work}$$

$$\vec{v} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$$ is an associated eigenvector.

For $\lambda = -1$

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = - \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 6a \quad\; - 6c &= 0 \\ 2a \quad\; - 2c &= 0 \\ 4a - 2b - 3c &= 0 \end{aligned} \qquad \begin{aligned} a &= 2 \\ b &= 1 \\ c &= 2 \end{aligned} \quad \text{works}$$

$\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is an associated eigenvector.

For $\lambda = 1$

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{aligned} 4a \quad\; - 6c &= 0 \\ 2a - 2b - 2c &= 0 \\ 4a - 2b - 5c &= 0 \end{aligned} \qquad \begin{aligned} a &= 6 \\ b &= 2 \\ c &= 4 \end{aligned}$$

$\vec{v} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$ is an associated eigenvector.

So, we get the general solution:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} e^{t}}$$

5.2.39 Find the general solution of the system:

$$\vec{x}\,' = \begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \vec{x}$$

Finding the eigenvalues:

$$\begin{vmatrix} -2-\lambda & 0 & 0 & 9 \\ 4 & 2-\lambda & 0 & -10 \\ 0 & 0 & -1-\lambda & 8 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left[(2-\lambda)(-1-\lambda)(1-\lambda)\right]$$

$$= (-2-\lambda)(2-\lambda)(-1-\lambda)(1-\lambda)$$

So, we have eigenvalues $\lambda = 1, -1, 2, -2$
with associated eigenvectors:

$\lambda = 1$

$$\begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a = 9$
$b = -8$ works, so $\vec{v} = \begin{pmatrix} 9 \\ -8 \\ 16 \\ 2 \end{pmatrix}$
$c = 16$
$d = 2$

$\lambda = -1$

$$\begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = - \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a = 0$
$b = 0$ works so $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
$c = 1$
$d = 0$

$\lambda = 2$

$$\begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a = 0$
$b = 1$ works so $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
$c = 0$
$d = 0$

$\lambda = -2$

$$\begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = -2 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a = 1$
$b = 0$ works so $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.
$c = 0$
$d = 0$

And we have the general solution:

$$\boxed{\vec{x}(t) = c_1\begin{pmatrix} 9 \\ -8 \\ 16 \\ 2 \end{pmatrix} e^{t} + c_2\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_4\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-2t}}$$

5.3.1 Find the two natural frequencies of the given system and describe its two natural modes of oscillation.

$m_1 = m_2 = 1$ $\qquad$ $k_1 = 0,\ k_2 = 2,\ k_3 = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \quad \vec{x}'' = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\vec{x}$$

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 - 4 = \lambda^2 + 4\lambda$$

$$\lambda = 0, -4$$

with associated eigenvectors:

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} -2a + 2b = 0 \\ 2a - 2b = 0 \end{matrix} \qquad a = b \text{ so } \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

$\lambda = -4$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -4 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{matrix} 2a + 2b = 0 \\ 2a + 2b = 0 \end{matrix} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works.}$$

So, the solution to our ODE is:

$$\cancel{\boxed{\vec{x}(t) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t) + b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t)}}$$

$$\boxed{\vec{x}(t) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b_1 t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t) + b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t)}$$

The natural ~~frequencise~~ frequencies are $\omega = 0$ and $\omega = 2$. In the $\omega = 0$ frequency it moves linearly in ~~and~~ one direction, while at $\omega = 2$ it oscillates in opposite directions with angular frequency $\omega = 2$

5.3.3 $m_1 = 1 \quad k_1 = 1$

$m_2 = 2 \quad k_2 = k_3 = 2$

So, we have the system of ODEs:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{x}'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -4 \end{pmatrix} \vec{x} = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \vec{x}$$

we get eigenvalues:

$$\begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = (3+\lambda)(2+\lambda) - 2$$

$$= 6 + 5\lambda + \lambda^2 - 2 = \lambda^2 + 5\lambda + 4$$

$\lambda = -1, -4$ with eigenvectors

$$\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$\cancel{-4a + 2b = 0}$ $\quad -2a + 2b = 0 \quad a = b$

$\cancel{a - 3b = 0}$ $\quad a - b = 0$

So,

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works.

$$\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -4\begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow \quad a + 2b = 0 \qquad 2a = -b$

$\quad\;\; a + 2b = 0$

So, $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ works.

So, the general solution is:

$$\boxed{\vec{x}(t) = a_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t + b_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t + a_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}\cos(2t) + b_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}\sin(2t)}$$

The two natural frequencies are $\omega = 1$ and $\omega = 2$. At $\omega = 2$ they oscillate in opposite directions with m_2 moving twice as far as m_1. At $\omega = 1$ they oscillate in the same direction with the same amplitude.

5.3.5. $m_1 = m_2 = 1 \qquad k_1 = 2, \; k_2 = 1, \; k_3 = 2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{x}'' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\vec{x}$$

this has eigenvalues:

$$\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (3+\lambda)^2 - 1 = \lambda^2 + 6\lambda + 8$$
$$= (\lambda+4)(\lambda+2)$$

So, we have eigenvalues $\lambda = -2, -4$.

The associated eigenvectors are:

$$\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -2\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{matrix} -a + b = 0 \\ a - b = 0 \end{matrix} \Rightarrow a = b \qquad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

$$\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -4\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{matrix} a + b = 0 \\ a + b = 0 \end{matrix} \Rightarrow a = -b \qquad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works.}$$

So, we get the general solution:

$$\boxed{\vec{x}(t) = a_1\begin{pmatrix}1\\1\end{pmatrix}\cos(\sqrt{2}\,t) + b_1\begin{pmatrix}1\\1\end{pmatrix}\sin(\sqrt{2}\,t) + a_2\begin{pmatrix}1\\-1\end{pmatrix}\cos(2t) + b_2\begin{pmatrix}1\\-1\end{pmatrix}\sin(2t)}$$

So, the natural frequencies are $\omega=\sqrt{2}$ and $\omega=2$. At $\omega=\sqrt{2}$ the masses oscillate in the same direction with the same ~~frequency~~ amplitude. At $\omega=2$ the masses oscillate in opposite directions but with the same amplitude.

5.3.9 The mass and spring system in problem 3 is set in motion from rest $(x_1'(0)=x_2'(0)=0)$ in its equilibrium position $(x_1(0)=x_2(0)=0)$ with the external forces $F_1(t)=0$ and $F_2(t)=120\cos(3t)$ acting on the masses m_1 and m_2, respectively. Find the resulting motion and describe it as a superposition of oscillations at three different frequencies

We get the system of ODEs:

$$\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\vec{x}'' = \begin{pmatrix}-3 & 2\\2 & -4\end{pmatrix}\vec{x} + \begin{pmatrix}0\\120\end{pmatrix}\cos(3t)$$

$$\Rightarrow \vec{x}'' = \begin{pmatrix}-3 & 2\\1 & -2\end{pmatrix}\vec{x} + \begin{pmatrix}0\\60\end{pmatrix}\cos(3t)$$

a particular solution to this ODE ~~is~~:

$$\vec{x}_p = \begin{pmatrix}c_1\\c_2\end{pmatrix}\cos(3t)$$

$$\vec{x}_p'' = -9\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\cos(3t)$$

$$= \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\cos(3t) + \begin{pmatrix} 0 \\ 60 \end{pmatrix}\cos(3t)$$

So,

$$-9c_1 = -3c_1 + 2c_2$$
$$-9c_2 = c_1 - 2c_2 + 60$$

$$\Rightarrow \quad 6c_1 + 2c_2 = 0$$
$$c_1 + 7c_2 + 60 = 0$$

$$\Rightarrow -\tfrac{1}{3}c_2 + 7c_2 + 60 = 0$$
$$\Rightarrow \tfrac{20}{3}c_2 = -60 \Rightarrow c_2 = -9 \qquad c_1 = 3$$

So,

$$\vec{x}(t) = a_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(t) + b_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin(t) + a_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}\cos(2t) + b_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}\sin(2t) + \begin{pmatrix} 3 \\ -9 \end{pmatrix}\cos(3t)$$

This is a superposition of oscillations with frequencies $\omega = 1$, $\omega = 2$, and $\omega = 3$.

Now, using our initial conditions

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 + 3 \\ a_1 - 2a_2 - 9 \end{pmatrix} \Rightarrow \cancel{a_1 = 3,\ a_2 = -6}\quad a_1 = 1,\ a_2 = -4$$

and

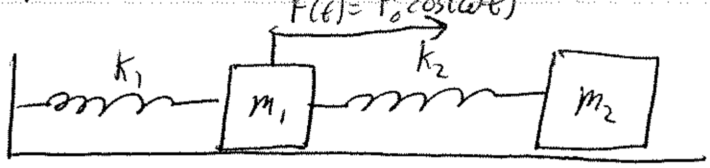
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 + 2b_2 \\ b_1 - 4b_2 \end{pmatrix} \Rightarrow b_1 = b_2 = 0.$$

So, our final solution is:

$$\cancel{\vec{x}(t) = 3\begin{pmatrix}1\\1\end{pmatrix}\cos(t) - 6\begin{pmatrix}1\\2\end{pmatrix}\cos(2t) + \begin{pmatrix}3\\9\end{pmatrix}\cos(3t)}$$

$$\boxed{\vec{x}(t) = \begin{pmatrix}1\\1\end{pmatrix}\cos(t) - 4\begin{pmatrix}1\\-2\end{pmatrix}\cos(2t) + \begin{pmatrix}3\\-9\end{pmatrix}\cos(3t)}$$

5.3.14 In the system below assume that $m_1 = 1$, $k_1 = 50$, $k_2 = 10$, and $F_0 = 5$ in mks units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!). Thus the effect of the second mass-and-spring pair will be to neutralize the effect of ~~this~~ the force on the first mass. This is an example of a dynamic damper. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.



We want to find the appropriate particular solution $\vec{x}_p$:

$$\vec{x}_p = \begin{pmatrix}c_1\\c_2\end{pmatrix}\cos(10t)$$

for the ODE:

$$\begin{pmatrix} 1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -60 & 10 \\ 10 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} \cos(10t)$$

$$\Rightarrow \vec{x}'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} \begin{pmatrix} -60 & 10 \\ 10 & -10 \end{pmatrix} \vec{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} \cos(10t)$$

$$\cancel{\Rightarrow \vec{x}'' = \Big(-60 + \frac{10}{m_2}}$$

$$\Rightarrow \vec{x}'' = \begin{pmatrix} -60 & 10 \\ \frac{10}{m_2} & -\frac{10}{m_2} \end{pmatrix} \vec{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} \cos(10t)$$

So, plugging in $\vec{x}_p$:

$$-100 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cos(10t) = \begin{pmatrix} -60 & 10 \\ \frac{10}{m_2} & \frac{-10}{m_2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cos(10t) + \begin{pmatrix} 5 \\ 0 \end{pmatrix} \cos(10t)$$

So, we get the relations:

$$-100 c_1 = -60 c_1 + 10 c_2 + 5$$
$$-100 c_2 = \frac{10}{m_2} c_1 - \frac{10}{m_2} c_2$$

Now, we want $c_1 = 0$. If this is the case then $c_2 = -\frac{1}{2}$ and $\boxed{m_2 = .1}$

So, for $c_1 = 0$ (x_p has no movement of m_1) then $m_2 = .1$.

5.4.1. Find a general solution to the system of equations and graph (for problem 1) the direction field and corresponding solution curves.

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix} \vec{x}$$

The system has eigenvalues:

$$\begin{vmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{vmatrix} = (2+\lambda)(4+\lambda)+1$$
$$= \lambda^2+6\lambda+9 = (\lambda+3)^2$$

So, we have eigenvalue $\lambda=-3$.

We want a vector $\vec{v}_2$ such that:

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}^2 \vec{v}_2 = 0$$

but

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \vec{v}_2 = \vec{v}_1 \neq 0.$$

Now,

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, if we take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

we get

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and the corresponding solutions:

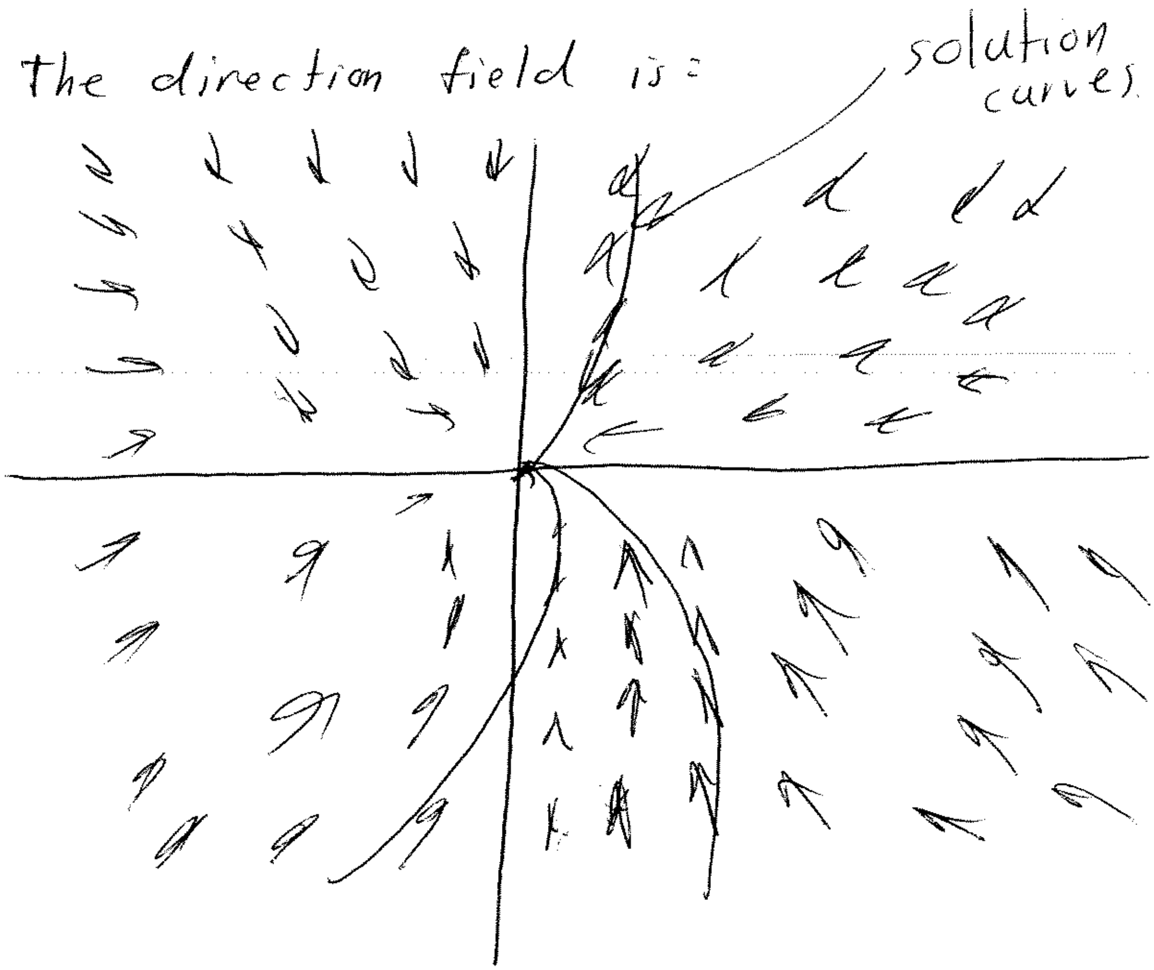
$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$

$$\vec{x}_2(t) = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-3t}$$

So, the general solution is:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-3t}}$$

The direction field is:



5.4.8.

$$\vec{x}' = \begin{pmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{pmatrix} \vec{x}$$

the characteristic equation is:

$$(13-\lambda)\left[(25-\lambda)(-5-\lambda) - (12)(-18)\right]$$

$$= (13-\lambda)\left(\lambda^2 - 20\lambda + 91\right)$$

$$= -(\lambda-13)^2(\lambda-7)$$ So, eigenvalues: $\lambda = 7$, $\lambda = 13$

For $\lambda = 7$ we have the eigenvector:

$$\begin{pmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 7 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 18a + 12b = 0 \\ -18a - 12b = 0 \\ 6a + 6b + 6c = 0 \end{array} \qquad \begin{array}{l} a = 2 \\ b = -3 \\ c = 1 \end{array} \quad \text{works.}$$

$$\vec{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

~~as for $\lambda = 13$ we must find two vectors satisfying:~~

$$\cancel{(A - \lambda I)^2 \vec{v}_2 = 0}$$

$$\cancel{(A - \lambda I)\vec{v}_2 = \vec{v}_1 \neq 0.}$$

Now, for $\lambda = 13$ we have eigenvectors

$$\begin{pmatrix} 12 & 12 & 0 \\ -18 & -18 & 0 \\ 6 & 6 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

both work! So, our general solution is:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{13t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{13t}}$$

5.4.15

$$\vec{x}\,' = \begin{pmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -2-\lambda & -9 & 0 \\ 1 & 4-\lambda & 0 \\ 1 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(-2-\lambda)(4-\lambda) + 9\right]$$

$$= (1-\lambda)(\lambda^2 - 2\lambda + 1)$$

$$= (1-\lambda)(\lambda-1)^2 = -(\lambda-1)^3$$

So, we have the eigenvalue $\lambda = 1$ which has order 3.

So, we have the eigenvector equation:

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we get two linearly independent eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for example}$$

Now, one must be "defective" and I'll bet it's the first. So, we get:

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, any vector works as long as

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \vec{v}_2 \neq 0.$$

Take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then

$$\vec{v}_1 = \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

So, we get the general solution:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] e^t + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t}$$

; 4.25

$$\vec{x}' = \begin{pmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix} \vec{x}$$

with eigenvalues (given in problem) $\lambda = 2, 2, 2$.

So, we have the eigenvector equation:

$$(A - \lambda I) = \begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Eigenvectors that work are:

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and that's it!

So, we take:

$$\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & 1 \end{pmatrix} = (A - \lambda I)^2$$

Now,

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & 1 \end{pmatrix} \begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, any <u>vector</u> $\vec{v}_3$ that satisfies

$$(A - \lambda I)^2 \vec{v}_3 \neq 0 \quad \cancel{\text{and } (A - \lambda I}\,)$$

will work. Take $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

we then get:

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \vec{v}_1$$

and

$$\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} = \vec{v}_2$$

So, we get solutions:

$$\vec{x}_1(t) = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} e^{2t}$$

$$\vec{x}_2(t) = \left[\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} \right] e^{2t}$$

$$\vec{x}_3(t) = \left[\frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] e^{2t}$$

So, our general solution is:

$$\vec{x}(t) = c_1\begin{pmatrix}-1\\0\\-1\end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix}-1\\0\\-1\end{pmatrix}t + \begin{pmatrix}-4\\-1\\0\end{pmatrix}\right]e^{2t} + c_3\left[\frac{1}{2}\begin{pmatrix}-1\\0\\-1\end{pmatrix}t^2 + \begin{pmatrix}-4\\-1\\0\end{pmatrix}t + \begin{pmatrix}1\\0\\0\end{pmatrix}\right]e^{2t}$$

5.4.33 The characteristic equation of the coefficient matrix A of the system

$$\vec{x}' = \begin{pmatrix}3 & -4 & 1 & 0\\4 & 3 & 0 & 1\\0 & 0 & 3 & -4\\0 & 0 & 4 & 3\end{pmatrix}\vec{x}$$

is

$$\phi(\lambda) = (\lambda^2 - 6\lambda + 25)^2 = 0$$

Therefore, A has the repeated complex conjugate pair $3 \pm 4i$ of eigenvalues. First show that the complex vectors

$$\vec{v}_1 = \begin{pmatrix}1\\i\\0\\0\end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix}0\\0\\1\\i\end{pmatrix}$$

Note: There is a typo in the book! The top entry here should be 0, not 9!

form a length 2 chain $\{\vec{v}_1, \vec{v}_2\}$ associated with the eigenvalue $\lambda = 3-4i$. Then calculate the real and imaginary parts of the complex-valued solutions $\vec{v}_1 e^{\lambda t}$ and $(\vec{v}_1 t + \vec{v}_2)e^{\lambda t}$

to find four independent real-valued solutions of $\vec{x}' = A\vec{x}$.

First, we verify it's a chain:

$$(A-\lambda I) = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix}$$

$$(A-\lambda I)\vec{v}_2 = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = \vec{v}_1$$

$$(A-\lambda I)\vec{v}_1 = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix}\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, it's a chain.

So, we have solutions:

$$\vec{x}_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} e^{(3-4i)t} \quad \vec{x}_2 = \left[\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}\right] e^{(3-4i)t}$$

Breaking this up into its real and complex parts we get:

$$e^{(3-4i)t} = e^{3t}\left(\cos(4t) - i\sin(4t)\right)$$

$$Re(\vec{x}_1) = \begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t}$$

$$Im(\vec{x}_1) = \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t}$$

$$Re(\vec{x}_2) = \begin{pmatrix} t\cos(4t) \\ t\sin(4t) \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t}$$

$$Im(\vec{x}_2) = \begin{pmatrix} -t\sin(4t) \\ t\cos(4t) \\ -\sin(4t) \\ \cos(4t) \end{pmatrix} e^{3t}$$

So, our final solution is:

$$\boxed{\begin{aligned}\vec{X}(t) = {} & c_1 \begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} \\ & + c_3 \begin{pmatrix} t\cos(4t) \\ t\sin(4t) \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t} + c_4 \begin{pmatrix} -t\sin(4t) \\ t\cos(4t) \\ -\sin(4t) \\ \cos(4t) \end{pmatrix}\end{aligned}}$$