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Solution Set #7

Transform the given equation or system into an equivalent system of first-order differential equations.

4.1.1. $x'' + 3x' + 7x = t^2$

Equivalent System:

~~$x_1 = x$~~
~~$x_2 = x_1'$~~
~~$x_2 = -3x_1 - 7x + t^2$~~

$$\boxed{\begin{aligned} x_1 &= x' \\ x_1' &= -3x_1 - 7x + t^2 \end{aligned}}$$

4.1.3. $t^2 x'' + t x' + (t^2 - 1)x = 0$

~~$x_1 = x'$~~
~~$x_2 = x_1'$~~
~~$t^2 x_2 = -t x_1 - (t^2 - 1)x$~~

$$\boxed{\begin{aligned} x_1 &= x' \\ t^2 x_1' &= -t x_1 - (t^2 - 1)x \end{aligned}}$$

4.1.13 Find a general solution to the ODE or system of ODEs, and use a graphing calculator to construct a direction field and typical solution curves

$$x' = -2y \qquad x(0) = 1$$
$$y' = 2x \qquad y(0) = 0$$

$$x'' = -2y' = -4x$$

$$\Rightarrow x'' + 4x = 0$$

$$\Rightarrow x(t) = A\cos(2t) + B\sin(2t)$$

and

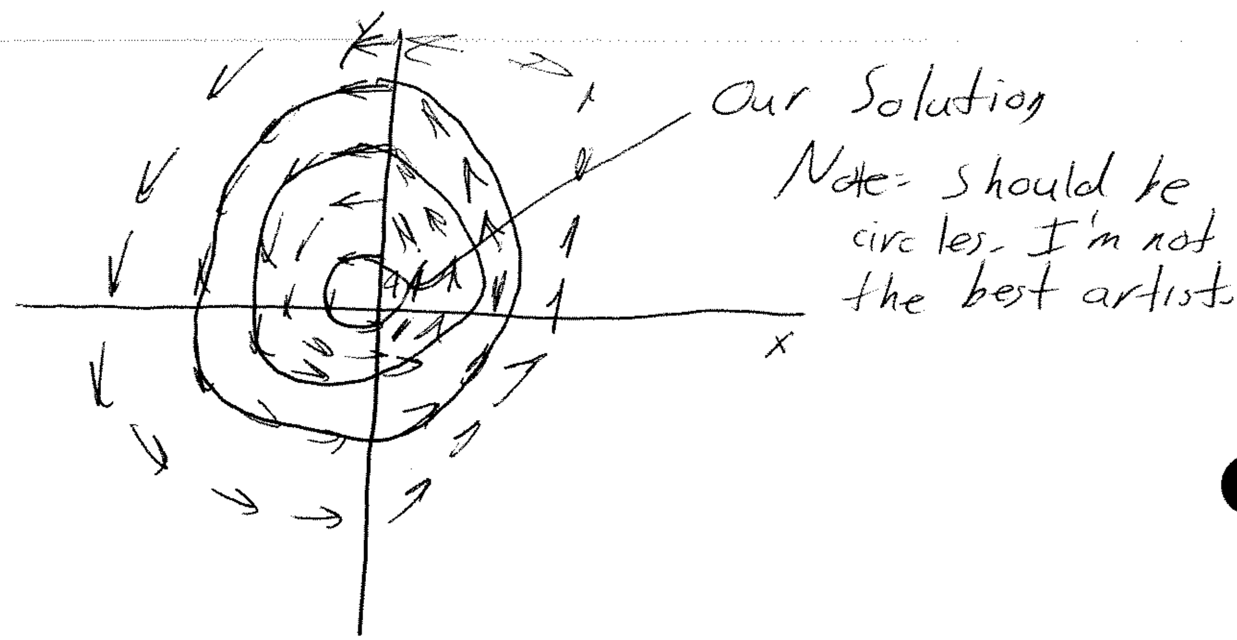
$$y = -\tfrac{1}{2}x'(t) = -\tfrac{1}{2}(-2A\sin(2t) + 2B\cos(2t))$$
$$= A\sin(2t) - B\cos(2t)$$

$$\Rightarrow y(t) = A\sin(2t) - B\cos(2t)$$

Now, solving for A and B:

$$x(0) = A = 1$$
$$y(0) = -B = 0$$
$$\Rightarrow x(t) = \cos(2t)$$
$$y(t) = \sin(2t)$$

Direction Field



7.1.22. a) Beginning with the general solution of the system $x'=-2y$, $y'=2x$ of problem 13, calculate x^2+y^2 to show that the trajectories are circles.

$$x(t) = A\cos(2t) + B\sin(2t)$$
$$y(t) = A\sin(2t) - B\cos(2t)$$

$$x^2(t)+y^2(t) = A^2\cos^2(2t) + 2AB\sin(2t)\cos(2t) + B^2\sin^2(2t) + A^2\sin^2(2t) - 2AB\sin(2t)\cos(2t) + B^2\cos^2(2t)$$

$$= A^2+B^2$$

So, the trajectory is a circle of radius A^2+B^2, where $A=x(0)$ and $B=-y(0)$

b) Show similarly that the trajectories of the system $x'=\frac{1}{2}y$, $y'=-8x$ of problem 15 are ellipses with equations of the form $16x^2+y^2=C^2$.

$$x'=\tfrac{1}{2}y, \quad y'=-8x$$

has solution:

$$x''=\tfrac{1}{2}y'=-4x \Rightarrow x''+4x=0$$

So, $x(t)=A\cos(2t)+B\sin(2t)$ and

$$y=2x'(t)=-4A\sin(2t)+4B\cos(2t)$$

So,

$$x(t)^2 = A^2\cos(2t) + 2AB\cos(2t)\sin(2t) + B^2\sin^2(2t)$$
$$y(t)^2 = 16A^2\sin^2(2t) - 16AB\cos(2t)\sin(2t) + 16B^2\cos^2(2t)$$

which gives us the relation:

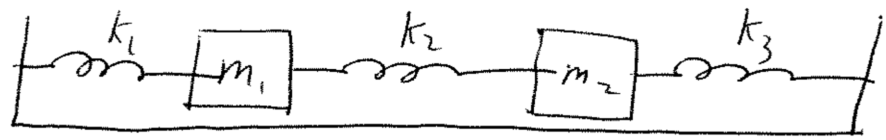
$$16x^2(t)+y^2(t) = 16(A^2+B^2) = C^2.$$

An ellipse.

4.1.24 Derive the equations

$$m_1 x_1'' = -(k_1+k_2)x_1 + k_2 x_2$$
$$m_2 x_2'' = k_2 x_1 - (k_2+k_3)x_2$$

for the displacements (from equilibrium) of the two masses:



If we view x_1 as the displacement of m_1 from equilibrium and x_2 as the displacement of m_2 from equilibrium then Hooke's law tells us:

$$F_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

and

$$F_2 = -k_2(x_2 - x_1) - k_3 x_2$$

and Newton's second law, $F=ma$, tells us:

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2$$

or, in a more compact form,

$$\boxed{\begin{aligned} m_1 x_1'' &= -(k_1+k_2)x_1 + k_2 x_2 \\ m_2 x_2'' &= k_2 x_1 - (k_2+k_3)x_2 \end{aligned}}$$

1.2.1 Find the general solution and construct a direction field and draw some typical solutions.

$$x' = -x + 3y$$
$$y' = 2y$$

Well, $y' = 2y$ has solution $\boxed{y(t) = Ae^{2t}}$

So,

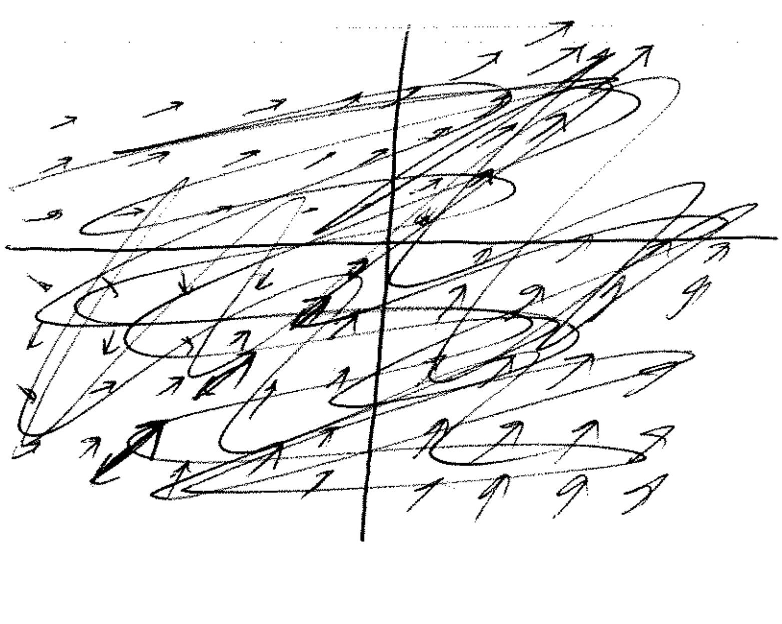
$$x' = -x + 3Ae^{2t}$$
$$\Rightarrow \quad x' + x = 3Ae^{2t}$$

which has integration factor $p(x) = e^{\int dx} = e^{x}$

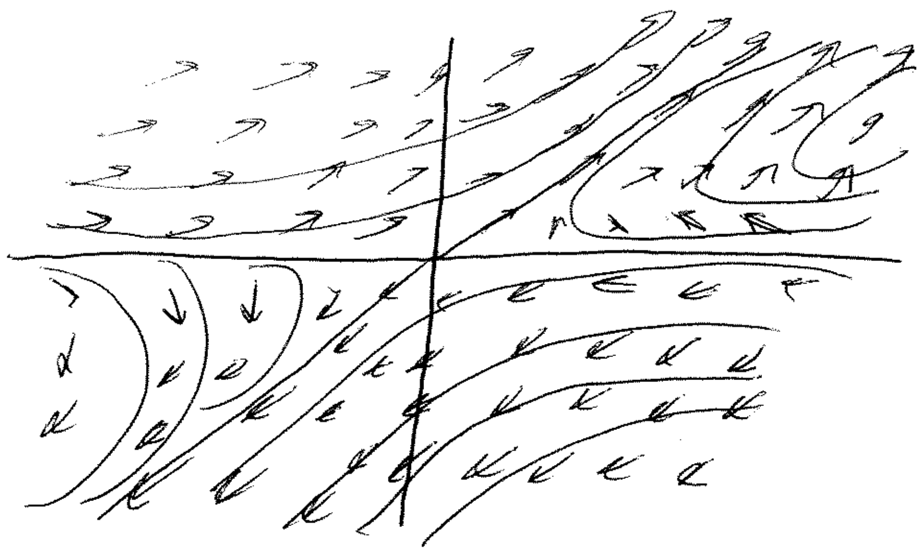
So,

$$\frac{d}{dt}(e^{t}x) = 3Ae^{3t}$$
$$\Rightarrow \quad e^{t}x = \frac{3A}{3}e^{3t} + B \Rightarrow \boxed{x(t) = \cancel{3}Ae^{2t} + Be^{-t}}$$

The direction field looks like:



Sorry.



4.2.10

$$x' + 2y' = 4x + 5y \qquad x(0) = 1$$
$$2x' - y' = 3x \qquad y(0) = -1$$

$$\Rightarrow \quad 5x' = 10x + 5y$$
$$-5y' = -5x - 10y \quad \Rightarrow \quad 5y' = 5x + 10y$$

So,

$$5x'' = 10x' + 5y' = 10x' + (5x + 10y)$$

$$\Rightarrow \quad 5x'' = 10x' + (5x + 10x' - 20x)$$
$$\Rightarrow \quad 5x'' = 20x' - 15x$$
$$\Rightarrow \quad x'' = 4x' - 3x \qquad \text{So,}$$
$$x'' - 4x' + 3x = 0$$

has characteristic equation

$$r^2 - 4r + 3 = (r-3)(r-1)$$

which has roots $r = 3, 1$

$$x(t) = c_1 e^{3t} + c_2 e^{t}$$

and

$$y' = 2x' - 3x = 3c_1 e^{3t} - c_2 e^{t}$$

$$\Rightarrow \quad y(t) = c_1 e^{3t} - c_2 e^{t}$$

Which has particular solution:

$$x(0) = c_1 + c_2 = 1$$
$$y(0) = c_1 - c_2 = -1$$

$$\Rightarrow \quad 2c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$
$$c_2 = 1$$

So,
$$\boxed{\begin{aligned} x(t) &= e^t \\ y(t) &= -e^t \end{aligned}}$$

4.2-19

$$x' = 4x - 2y$$
$$y' = -4x + 4y - 2z$$
$$z' = \quad -4y + 4z$$

$$x'' = 4x' - 2y' = 4x' - 2(-4x + 4y - 2z)$$

$$x'' = 4x' + 8x - 8y + 4z$$

$$\Rightarrow x^{(3)} = 4x'' + 8x' - 8y' + 4z' = 4x'' + 8x' - 8y' + 4(-4y + 4z)$$

$$\Rightarrow x^{(3)} = 4x'' + 8x' - 8y' - 16y + 16z$$
$$\Rightarrow x^{(3)} = 4x'' + 8x' - 8y' - 16y + 8(-y' - 4x + 4y)$$
$$\Rightarrow x^{(3)} = 4x'' + 8x' - 16y' + 16y - 32x$$
$$\Rightarrow x^{(3)} = 4x'' + 8x' - 8(4x' - x'') + 8(4x - x') - 32x$$
$$\Rightarrow x^{(3)} = 12x'' - 32x'$$
$$\Rightarrow \quad x^{(3)} - 12x'' + 32x' = 0$$

the characteristic equation is:

$$r(r-8)(r-4)$$

with roots $r = 0, 4, 8$.

So,

$$x(t) = c_1 + c_2 e^{4t} + c_3 e^{8t}$$

$$x'(t) = 4c_2 e^{4t} + 8c_3 e^{8t}$$

$$y(t) = 2x - \tfrac{1}{2}x' = 2c_1 - 2c_3 e^{8t}$$

and

$$\cancel{2}\ z(t) = -2x(t) + 2y(t) - \tfrac{1}{2}y'(t)$$

$$= -2c_1 - 2c_2 e^{4t} - 2c_3 e^{8t} + 4c_1 - 4c_3 e^{8t} + 8c_3 e^{8t}$$

$$\Rightarrow z(t) = 2c_1 - 2c_2 e^{4t} + 2c_3 e^{8t}$$

So,

$$\boxed{\begin{aligned} x(t) &= c_1 + c_2 e^{4t} + c_3 e^{8t} \\ y(t) &= 2c_1 - 2c_3 e^{8t} \\ z(t) &= 2c_1 - 2c_2 e^{4t} + 2c_3 e^{8t} \end{aligned}}$$

4.2.28 Calculate the operational determinant of the given system in order to determine how many arbitrary constants should appear in a general solution. Then solve the system explicitly to find a general solution, if possible.

~~opera~~

$$(D^2+D)x + D^2 y = 2e^{-t}$$
$$(D^2-1)x + (D^2-D)y = 0$$

Operational determinant:

$$(D^2+D)(D^2-D) - D^2(D^2-1) = D^4 - D^3 + D^3 - D^2 - D^4 + D^2 = 0$$

So, there are 0 (!) arbitrary constants. How is this possible? Well, if we look at our system we get the relation:

$$(D+1)x + Dy = 2e^{-t}$$

$$\Rightarrow Dy = 2e^{-t} - (D+1)x$$
$$\Rightarrow D^2y = -2e^{-t} - D^2x$$
$$\Rightarrow D^2x + D^2y = -2e^{-t}$$

Plugging this into the ~~first~~ ~~second~~ first relation we get

$$Dx - 2e^{-t} = 2e^{-t}$$

$$\Rightarrow Dx = 4e^{-t} \quad \Rightarrow \quad x(t) = -4e^{-t} + C_1$$

the second relation gives us

$$-2e^{-t} - (-4e^{-t} + C_1) - Dy = 0$$

$$\Rightarrow Dy = 2e^{-t} - C_1$$

$$\Rightarrow y(t) = -2e^{-t} - C_1 t + C_2$$

$\cancel{D^2x = -4e^{-t}}$
$\cancel{Dx = 4e^{-t}}$
$\cancel{D^2y = -2e^{-t}}$
$\cancel{Dy = 2e^{-t}}\ C_1$

$\cancel{-4e^{-t} + 4e^{-t}}$

However, this cannot be, as $D^2x + D^2y = -6e^{-t}$ which contradicts our earlier relation. So, there is no solution.

4.2-30 Suppose that the salt concentration in each of the two brine tanks of example 2 of section 4.1 initially ($t=0$) is 0.5 lb/gal. Then solve the system in Eq. 5 there to find the amounts $x(t)$ and $y(t)$ of salt in the two tanks at time t.

The system in Eq. 5 is:

$$20x' = -6x + y$$
$$20y' = 6x - 3y$$

$$\Rightarrow \quad 20x'' = -6x' + y'$$
$$\Rightarrow \quad 400x'' = -120x' + 20y'$$
$$\Rightarrow \quad 400x'' = -120x' + 6x - 3y$$
$$\Rightarrow \quad 400x'' = -120x' + 6x - 3(20x' + 6x)$$
$$\Rightarrow \quad 400x'' = -120x' + 6x - 60x' - 18x$$
$$\Rightarrow \quad 400x'' + 180x' + 12x = 0$$

which has characteristic equation:

$$400r^2 + 180r + 12 = 0$$

$$r = \frac{-180 \pm \sqrt{180^2 - 4(400)(12)}}{2(400)}$$

$$= -\frac{9}{40} \pm \frac{\sqrt{33}}{40} = \frac{-9 \pm \sqrt{33}}{40}$$

So,

$$x(t) = c_1 e^{-\frac{9+\sqrt{33}}{40}t} + c_2 e^{-\frac{9-\sqrt{33}}{40}t}$$

$$y(t) = 20x' + 6x$$

$\Rightarrow$

$$y(t) = c_1\left(\frac{-9+\sqrt{33}}{2}\right)e^{-\frac{9+\sqrt{33}}{40}t} + c_2\left(\frac{-9-\sqrt{33}}{2}\right)e^{\frac{-9-\sqrt{33}}{40}t}$$

$$+ 6c_1 e^{-\frac{9+\sqrt{33}}{40}t} + 6c_2 e^{\frac{-9-\sqrt{33}}{40}t}$$

$$= c_1\left(\frac{9+\sqrt{33}}{2}\right)e^{-\frac{9+\sqrt{33}}{40}t} + c_2\left(\frac{9-\sqrt{33}}{2}\right)e^{\frac{-9-\sqrt{33}}{40}t}$$

So, given initial conditions:

$$x(0) = c_1 + c_2 = .5$$
$$y(0) = c_1\left(\frac{9+\sqrt{33}}{2}\right) + c_2\left(\frac{9-\sqrt{33}}{2}\right) = .5$$

Solving this system we get

$$c_2\left(\frac{9-\sqrt{33}}{2}\right) - c_2\left(\frac{9+\sqrt{33}}{2}\right) = .5 - .5\left(\frac{9+\sqrt{33}}{2}\right)$$

$$\Rightarrow -\sqrt{33}\,c_2 = -\left(\frac{7+\sqrt{33}}{4}\right)$$

$$\Rightarrow c_2 = \frac{7+\sqrt{33}}{4\sqrt{33}} = \frac{33+7\sqrt{33}}{132}$$

and

$$c_1 = \frac{1}{2} - c_2 = \frac{+33 - 7\sqrt{33}}{132}.$$

So,

$$\boxed{x(t) = \left(\frac{33-7\sqrt{33}}{132}\right) e^{-\frac{9+\sqrt{33}}{40}t} + \left(\frac{33+7\sqrt{33}}{132}\right) e^{-\frac{9-\sqrt{33}}{40}t}}$$

and

$$y(t) = \left(\frac{33-7\sqrt{33}}{132}\right)\left(\frac{9+\sqrt{33}}{2}\right) e^{-\frac{9+\sqrt{33}}{40}t} + \left(\frac{33+7\sqrt{33}}{132}\right)\left(\frac{9-\sqrt{33}}{2}\right) e^{-\frac{9-\sqrt{33}}{40}}$$

$$= \frac{66-30\sqrt{33}}{264} e^{-\frac{9+\sqrt{33}}{40}t} + \frac{66+30\sqrt{33}}{264} e^{-\frac{9-\sqrt{33}}{40}t}$$

$$= \boxed{\frac{11-5\sqrt{33}}{44} e^{-\frac{9+\sqrt{33}}{40}t} + \frac{11+5\sqrt{33}}{44} e^{-\frac{9-\sqrt{33}}{40}t}}$$

I apologize for assigning this one!

5.1.1. Let

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix}$$

Find

a) $2A + 3B$

$$2\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} + 3\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -18 \\ 23 & 17 \end{pmatrix}$$

b) $3A - 2B$

$$3\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} - 2\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 19 \end{pmatrix}$$

c) AB

$$\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -9 & -11 \\ 47 & -9 \end{pmatrix}$$

d) BA

$$\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} -10 & -37 \\ 14 & -8 \end{pmatrix}$$

5.1-7 Compute the determinant of the matrices A and B and verify the results are consistent with:

$$\det(A)\det(B) = \det(AB)$$

$$\det(A) = 14 - (-12) = 26$$

$$\det(B) = 3 - (-20) = 23$$

$$\det(AB) = 81 - (-517) = 598 = 26 \times 23.$$

So, math is consistent. Phew!

5.1.15 Write the system in the form

$$\vec{x}'(t) = \vec{P}(t)\,\vec{x}(t) + \vec{f}(t)$$

$$x' = y + z$$
$$y' = z + x$$
$$z' = x + y$$

$$\Rightarrow \quad \vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$$

with $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

5.1.21. Verify the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution to the system

$$\vec{x}' = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \vec{x}$$

$$\vec{x}_1 = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} \qquad \vec{x}_1' = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}$$

So,

$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} = \vec{x}_1'$$

So, it checks out.

$$\vec{x}_2 = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} \qquad \vec{x}_2' = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix} = \vec{x}_2'$$

So, it checks out too.

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = +e^{3t} \neq 0.$$ So, linearly independent

General Solution:

$$\vec{x} = c_1\begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} + c_2\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

5.1.27

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$$

$$\vec{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{x}_1' = e^{2t}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e^{2t}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

So, it checks out.

$$\vec{x}_2 = e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \vec{x}_2' = e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So, it checks out.

$$\vec{x}_3 = e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{x}_3' = e^{-t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = e^{-t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

So, it checks out.

$$W(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 1 + 2 + 0 = 3 \neq 0$$

So, they are linearly independent

$$\boxed{\vec{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$$