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Zwick
Math 2280
1, 2, 4, 17, 24

Assignment # 6 Solutions

3.6.1 Express the solution of the given IVP as the sum of two oscillations as in Eq. (8). Graph the solution and label its period.

$x'' + 9x = 10\cos(2t) \qquad x(0) = x'(0) = 0.$

x_h can be found by solving the characteristic equation

$$r^2 + 9 = 0$$

$\Rightarrow r = \pm 3i.$

This gives solutions:

$$x_h = c_1 \cos(3t) + c_2 \sin(3t)$$

$x_p = A\cos(2t) + B\sin(2t)$ Guess the particular solution.

$x_p' = -2A\sin(2t) + 2B\cos(2t)$

$x_p'' = -4A\cos(2t) - 4B\sin(2t)$

So,

$$x_p'' + 9x_p = 5A\cos(2t) + 5B\sin(2t) = 10\cos(2t)$$

$\Rightarrow A = 2 \quad B = 0.$

So,

$$x_p = 2\cos(2t)$$

Now, solving for c_1 and c_2 we get:

$$x(0) = c_1 \cos(3(0)) + c_2 \sin(3(0)) + 2\cos(2(0))$$

$$= c_1 + 2 = 0 \quad \Rightarrow c_1 = -2$$

$$\Rightarrow x(t) = -2\cos(3t) + c_2 \sin(3t) + 2\cos(2t)$$

$$x'(t) = 6\sin(3t) + 3c_2 \cos(3t) - 4\sin(2t)$$

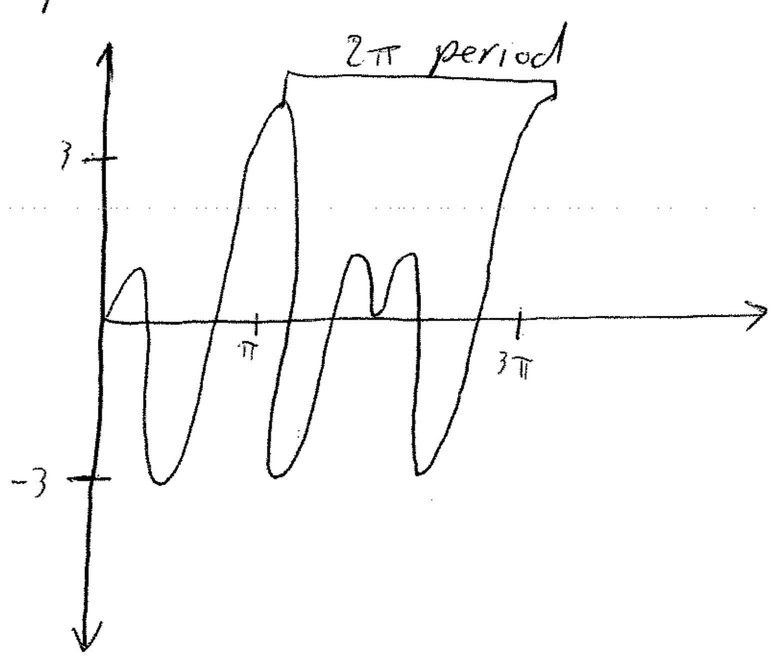
$$x'(0) = 3c_2 = 0 \quad \Rightarrow c_2 = 0$$

So,

$$x(t) = -2\cos(3t) + 2\cos(2t).$$

This is in the proper form.

Graph:



6.2. $x'' + 4x = 5\sin(3t)$ $\qquad x(0) = x'(0) = 0$

the homogenous solution is:

$$r^2 + 4 = 0 \qquad \Rightarrow r = \pm 2i$$

$$x_h = c_1 \cos(2t) + c_2 \sin(2t)$$

$$x_p = A\cos(3t) + B\sin(3t)$$
$$x_p' = -3A\sin(3t) + 3B\cos(3t)$$
$$x_p'' = -9A\cos(3t) - 9B\sin(3t)$$

$$x_p'' + 4x_p = -5A\cos(3t) - 5B\sin(3t) = 5\sin(3t)$$

$$A = 0 \quad B = -1$$

$$x_p = -\sin(3t)$$

So,
$$x(t) = c_1\cos(2t) + c_2\sin(2t) - \sin(3t)$$
$$x'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) - 3\cos(3t)$$

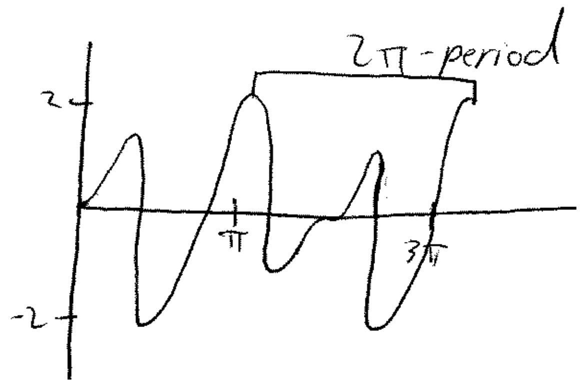
$$x(0) = c_1 = 0$$
$$x'(0) = +2c_2 - 3 = 0 \qquad c_2 = \tfrac{3}{2}$$

$$\Rightarrow \quad x(t) = \tfrac{3}{2}\sin(2t) - \sin(3t)$$

Converting to the "standard" form:

$$x(t) = \tfrac{3}{2}\cos\left(2t - \tfrac{\pi}{2}\right) - \cos\left(3t - \tfrac{\pi}{2}\right)$$

Graph:



3.6.9. Find x_{sp} and graph it compared to $F(t)/m\omega$.

$$2x''+2x'+x=3\sin(10t)$$

Now, x_{sp} is a particular solution to the ODE. It will have the form:

$$x_{sp}=A\cos(10t)+B\sin(10t)$$
$$x'_{sp}=-10A\sin(10t)+10B\cos(10t)$$
$$x''_{sp}=-100A\cos(10t)-100B\sin(10t)$$

So,
$$2x''_{sp}+2x'_{sp}+x_{sp}=3\sin(10t)$$

$$(-200A+20B+A)\cos(10t)+(-200B-20A+B)\sin(10t)$$

$$-199A+20B=0 \qquad A=\frac{20}{199}B \qquad A=\frac{-60}{40001}$$
$$-20A-199B=3$$

$$\frac{-400}{199}B-199B=3 \Rightarrow B=-\frac{3(199)}{40{,}001}=-\frac{597}{40001}$$

So,

$$x_{sp} = -\frac{1}{40001}\left(60\cos(10t) + 597\sin(10t)\right)$$

Now,

$$C = \sqrt{\left(\frac{-60}{40001}\right)^2 + \left(\frac{-597}{40001}\right)^2}$$

$$= \sqrt{\frac{360{,}009}{40{,}001}} = \boxed{\frac{3}{\sqrt{40{,}001}}}$$

and

$$\alpha = \tan^{-1}\left(\frac{\cancel{4}-597}{-60}\right) = \boxed{\pi + \tan^{-1}\left(\frac{199}{20}\right)}$$

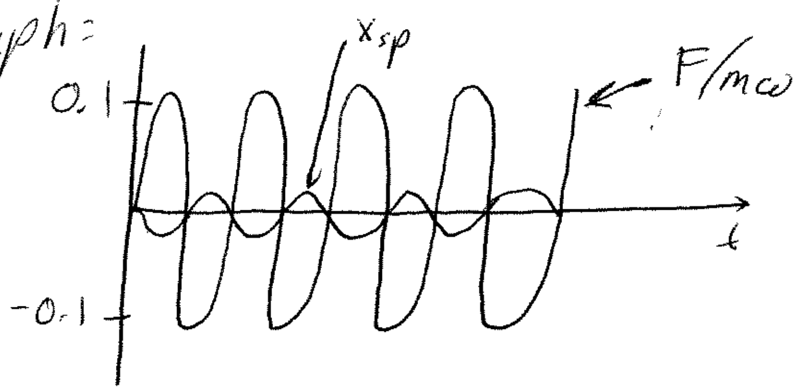
So,

$$x_{sp} = C\cos(10t - \alpha)$$

with

$$C = \frac{3}{\sqrt{40{,}001}} \qquad \alpha = \pi + \tan^{-1}\left(\frac{199}{20}\right)$$

Graph:



3.6.17. Using the equation for $C(\omega)$ from section 3-6:

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

We have:

$F_0 = 50 \quad m = 1 \quad c = 6 \quad k = 45$

So,

$$C(\omega) = \frac{50}{\sqrt{(45 - \omega^2)^2 + (6\omega)^2}}$$

We get practical resonance at a minimum for $(45 - \omega^2)^2 + (6\omega)^2$

$$(45 - \omega^2)^2 + (6\omega)^2 = \omega^4 - 54\omega^2 + 2025 = B(\omega)$$

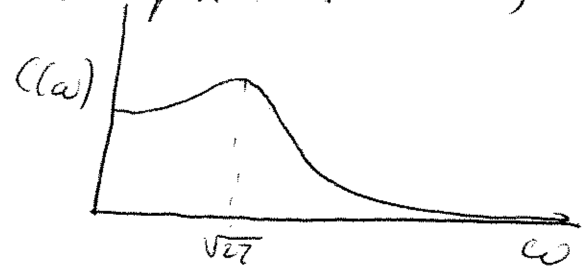
$$B'(\omega) = 4\omega^3 - 108\omega = (4\omega^2 - 108)\omega$$
$$B''(\omega) = 12\omega^2 - 108$$

$B'(\omega) = 0$ at $\omega = \pm\sqrt{27}, 0$

Local max at $\omega = 0$ local min at $\omega = \pm\sqrt{27}$

Practical Resonance at ~~$\omega = \sqrt{29}$~~ $\boxed{\omega = \sqrt{27}}$

Graph of $C(\omega)$



6.24. A mass on a spring without damping is acted on by an external force $F(t) = F_0 \cos^3(\omega t)$. Show that there are <u>two</u> values of ω at which resonance occurs

$$F_0 \cos^3 \omega t = \frac{F_0 \cos(\omega t) + F_0 \cos(\omega t)\cos(2\omega t)}{2}$$

using $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$\frac{F_0 \cos(\omega t) + F_0 \cos(\omega t)\cos(2\omega t)}{2}$$

$$= \frac{F_0 \cos(\omega t)}{2} + \frac{F_0 \cos(\omega t)}{4} + \frac{F_0 \cos(3\omega t)}{4}$$

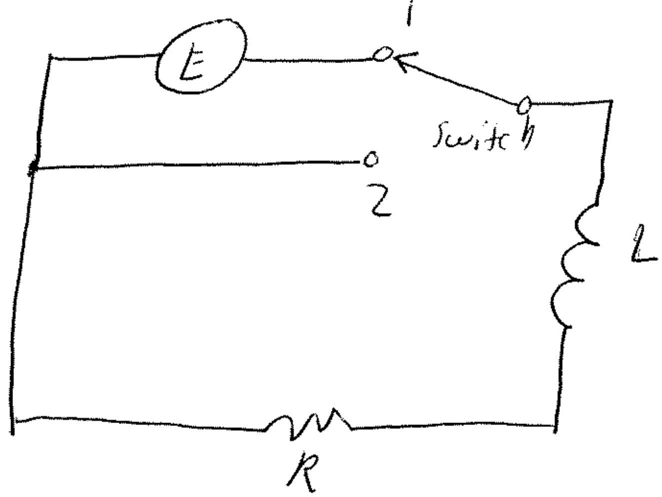
using $\cos(A)\cos(B) = \dfrac{\cos(A+B) + \cos(A-B)}{2}$

$$= \frac{3F_0 \cos(\omega t)}{4} + \frac{F_0 \cos(3\omega t)}{4}$$

So, ~~ω occu~~ resonance occurs when

$$\boxed{\sqrt{\frac{k}{m}} = \omega \text{ or } 3\omega}$$

3.7.1



~~It~~ In the circuit shown above suppose that $L = 5\,H$, $R = 25\,\Omega$, and the source E of EMF is a battery supplying 100V to the circuit. Suppose also that the switch has been in position 1 for a long time, so that a steady current of 4A is flowing in the circuit. At time $t=0$, the switch is thrown to position 2, so that $I(0) = 4$ and $E = 0$ for $t \geq 0$. Find $I(t)$.

The ODE is:

$$0 = L\frac{dI}{dt} + RI \qquad \text{with } I(0) = 4 \text{ Amps}, \; L = 5\,H, \; R = 25\,\Omega$$

So,

$$0 = 5\frac{dI}{dt} + 25I$$

$$\Rightarrow 0 = \frac{dI}{dt} + 5I \Rightarrow \frac{dI}{dt} = -5I$$

So,

$$I(t) = Ce^{-5t}$$

$I(0) = C = 4$, So, $\boxed{I(t) = 4e^{-5t} \text{ Amps}}$

3.7.5. In the same circuit, with the switch in position 1, suppose ~~L=2, R=6~~ $E(t)=100e^{-10t}\cos(60t)$, $R=20$, $L=2$, and $I(0)=0$. Find $I(t)$

We get the ODE:

$$100e^{-10t}\cos(60t) = 2\frac{dI}{dt} + 20I \qquad I(0)=0.$$

$$\Rightarrow \frac{dI}{dt} + 10I = 50e^{-10t}\cos(60t)$$

$$\rho(t) = e^{\int 10\,dt} = e^{10t}$$

$$\Rightarrow \frac{d(e^{10t}I)}{dt} = 50\cos(60t)$$

$$\Rightarrow e^{10t}I = \frac{5}{6}\sin(60t) + C$$

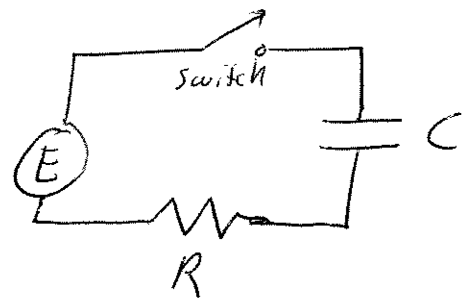
$$\Rightarrow I(t) = \frac{5}{6}e^{-10t}\sin(60t) + Ce^{-10t}$$

at $t=0$ we have $I(0) = C = 0$.

So,

$$\boxed{I(t) = \frac{5}{6}e^{-10t}\sin(60t)}$$

?. 7. 10



An EMF of ~~V(t)=~~ $E(t) = E_0 \cos(\omega t)$ is applied to the above circuit at $t=0$ (when the switch is closed.) and $Q(0)=0$. Substitute $Q_{sp}(t) = A\cos(\omega t) + B\sin(\omega t)$ in the ODE to show that

$$Q_{sp}(t) = \frac{E_0\, C}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\omega t - \beta)$$

where $\beta = \tan^{-1}(\omega R C)$

The ODE is:

$$E_0 \cos(\omega t) = R\frac{dQ_{sp}}{dt} + \frac{Q_{sp}}{C}$$

$$Q_{sp} = A\cos(\omega t) + B\sin(\omega t)$$
$$Q_{sp}' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$\Rightarrow E_0\cos(\omega t) = \left(-RA\omega + \frac{B}{C}\right)\sin(\omega t) + \left(RB\omega + \frac{A}{C}\right)\cos(\omega t)$$

$$\Rightarrow \quad -RA\omega + \frac{B}{C} = 0 \quad \Rightarrow (-\omega R C)A + B = 0$$
$$RB\omega + \frac{A}{C} = E_0 \qquad \Rightarrow B = \omega R C A$$

$$\Rightarrow \quad \omega^2 R^2 C A + A/C = E_0$$
$$\Rightarrow \quad A(1+\omega^2 R^2 C^2) = E_0 C \Rightarrow A = \frac{E_0\, C}{(1+\omega^2 R^2 C^2)}$$

So, we get

$$Q_{sp}(t) = \frac{E_0 C}{(1 + \omega^2 K^2 C^2)} \cos(\omega t) + \frac{\omega E_0 R C^2}{(1 + \omega^2 K^2 C^2)} \overset{\sin}{\cancel{\cos}}(\omega t)$$

$$= \frac{E_0 C}{\sqrt{1 + \omega^2 R^2 C^2}} \left(\frac{\cos(\omega t)}{\sqrt{1 + \omega^2 R^2 C^2}} + \frac{\omega R C \sin(\omega t)}{\sqrt{1 + \omega^2 K^2 C^2}} \right)$$

$$\cos(\omega t - \alpha) = \cos(\omega t)\cos(\alpha) + \sin(\omega t)\sin(\alpha)$$

So, $\cos(\alpha) = \dfrac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$ $\quad \sin(\alpha) = \dfrac{\omega R C}{\sqrt{1 + \omega^2 R^2 C^2}}$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{\omega R C / \sqrt{1 + \omega^2 R^2 C^2}}{1 / \sqrt{1 + \omega^2 R^2 C^2}} \right)$$

$$= \tan^{-1}(\omega R C).$$

So, that's that.

3.7.17 For an RLC circuit with input voltage $E(t)$ described below, find the current $I(t)$ using $Q(0)$ and $I(0)$.

$R = 16\,\Omega \quad L = 2H \quad C = 0.02F$
$E(t) = 100V \quad I(0) = 0 \quad Q(0) = 5$

$$E(t) = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \left(\frac{1}{F}\right)Q$$

$$\Rightarrow 100 = 2\frac{d^2Q}{dt^2} + 16\frac{dQ}{dt} + 50Q$$

$$\Rightarrow 50 = \frac{d^2Q}{dt^2} + 8\frac{dQ}{dt} + 25Q$$

~~Guess $Q(t) =$~~

Solve the homogenous equation's characteristic equation:

$$r^2 + 8r + 25 = 0$$

$$\frac{-8 \pm \sqrt{8^2 - 4(1)(25)}}{2} = -4 \pm \frac{6i}{2} = -4 \pm 3i$$

So,

$$Q(t) = e^{-4t}[c_1\cos(3t) + c_2\sin(3t)]$$

$$Q_{sp} = A \quad \Rightarrow \quad A = 2.$$

$$Q(t) = e^{-4t}[c_1\cos(3t) + c_2\sin(3t)] + 2$$
$$Q'(t) = e^{-4t}[-3c_1\sin(3t) + 3c_2\cos(3t)] - 4e^{-4t}[c_1\cos(3t) + c_2\sin(3t)]$$
$$Q(0) = c_1 + 2 = \cancel{0}\,5 \qquad c_1 = 3$$
$$Q'(0) = 3c_2 - 4c_1 = 0 \qquad c_2 = 4.$$

So,

$$I(t) = e^{-4t}[-9\sin(3t) + 12\cos(3t)] - 4e^{-4t}[3\cos(3t) + 4\sin(3t)]$$

$$= \boxed{-25e^{-4t}\sin(3t)}$$

3.7.19

$R = 60\,\Omega \quad L = 2H \quad C = .0025F;$
$E(t) = 100e^{-10t}V; \quad I(0) = 0; \quad Q(0) = 1$

$$\Rightarrow 100e^{-10t} = 2\frac{d^2Q}{dt^2} + 60\frac{dQ}{dt} + \frac{1}{.0025}Q$$

$$\Rightarrow 50e^{-10t} = \frac{d^2Q}{dt^2} + 30\frac{dQ}{dt} + 200Q$$

Solve the characteristic equation for the homogenous part:

$$r^2 + 30r + 200 = 0$$

$$r = \frac{-30 \pm \sqrt{(30)^2 - 4(1)(200)}}{2} = \frac{-30 \pm 10}{2} = -10, -20$$

So,

$$Q_h(t) = c_1 e^{-10t} + c_2 e^{-20t}$$

Now, the guess for Q_p would be

$$Q_p = Ae^{-10t}$$

but that's not LI from $Q_h(t)$ so, we guess:

$$Q_p = Ate^{-10t};$$

Plugging this in we get:

$$Q_p = Ate^{-10t}$$
$$Q_p' = -10Ate^{-10t} + Ae^{-10t}$$
$$Q_p'' = 100Ate^{-10t} - 20Ae^{-10t}.$$

So,

$$50e^{-10t} = (100Ate^{-10t} - 20Ae^{-10t}) + 30(-10Ate^{-10t} + Ae^{-10t}) + 200(Ate^{-10t})$$

$$\Rightarrow 50e^{-10t} = 10Ae^{-10t} \quad \Rightarrow A = 5.$$

So, we get:

$$Q(t) = c_1 e^{-10t} + c_2 e^{-20t} + 5te^{-10t}$$

$$Q'(t) = I(t) = -10c_1 e^{-10t} - 20c_2 e^{-20t} - 50te^{-10t} + 5e^{-10t}$$

$$Q(0) = c_1 + c_2 = 1 \qquad c_2 = 1 - c_1$$
$$I(0) = -10c_1 - 20c_2 + 5 = 0$$
$$\cancel{\Rightarrow -10(1} \Rightarrow -10c_1 - 20(1 - c_1) + 5 = 0$$
$$\Rightarrow 10c_1 - 15 = 0 \Rightarrow c_1 = \frac{3}{2} \quad c_2 = -\frac{1}{2}$$

So,

$$\boxed{I(t) = -10e^{-10t} + 10e^{-20t} - 50te^{-10t}}$$