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Math 2280

Assignment #5 Solutions

3.3.1 Find the general solutions of the DE.

$$y'' - 4y = 0$$

characteristic equation

$$r^2 - 4 = 0 \Rightarrow r = \pm 2$$

So,

$$\boxed{y(x) = c_1 e^{-2x} + c_2 e^{2x}}$$

3.3.10 $5y^{(4)} + 3y^{(3)} = 0$

characteristic equation

$$5r^4 + 3r^3 = 0$$

$$\Rightarrow (5r+3)r^3 \qquad r = \{0, -\tfrac{3}{5}\}$$

with 0 a root of order 3. So,

$$y(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 x^2 e^{0x} + c_4 e^{-3/5 x}$$

$$= \boxed{c_1 + c_2 x + c_3 x^2 + c_4 e^{-3/5 x}}$$

3.3.25 Find the solution to the initial value problem

$3y^{(3)}+2y''=0$; $y(0)=-1$, $y'(0)=0$, $y''(0)=1$

characteristic equation:

$3r^3+2r^2=0 \Rightarrow (3r+2)r^2=0$

So, $r=-\frac{2}{3}, 0$ 0 is a 2nd order root.

$y(x)= c_1+c_2x+c_3e^{-2/3x}$

$y(0)= c_1+c_3=-1$

$y'(x)= c_2-\frac{2}{3}c_3e^{-2/3x}$

$y'(0)= c_2-\frac{2}{3}c_3=0$

$y''(x)=\frac{4}{9}c_3e^{-2/3x}$

$y''(0)=\frac{4}{9}c_3=1$

So,

$c_3=\frac{9}{4}$ $c_2=\frac{3}{2}$ $c_1=-\frac{5}{4}$

$$\boxed{y(x)=-\frac{5}{4}+\frac{3}{2}x+\frac{9}{4}e^{-\frac{2}{3}x}}$$

3.3.30 Find general solutions for the ODE

$y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$

Characteristic Equation

$r^4 - r^3 + r^2 - 3r - 6 = 0$

$(-1)^4 - (-1)^3 + (-1)^2 - 3(-1) - 6 = 1 + 1 + 1 + 3 - 6 = 0$

So $r = -1$ is a root:

$(r+1)(r^3 - 2r^2 + 3r - 6)$

$r = 2$ works as a root for the second equation

$(r+1)(r-2)(r^2 \quad +3)$

$r = \pm\sqrt{3}\, i$ are the remaining 2 solutions

So, the general solution is:

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos(\sqrt{3}x) + c_4 \sin(\sqrt{3}x)}$$

3.3.43 a) Use Euler's formula to show that every complex number can be written in the form $re^{i\theta}$, where $r \geqq 0$ and $-\pi < \theta \leq \pi$.

$$z = x + iy$$

can be written as:

$$z = re^{i\theta} \quad \text{where}$$

$$r = \sqrt{x^2+y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

as $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$ $\quad \sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

for this choice of θ (taking $-\pi < \theta \leq \pi$) and so

$$re^{i\theta} = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{iy}{\sqrt{x^2+y^2}}\right) = x+iy = z$$

using $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula)

b) ~~#1~~ Express the numbers $4, -2, 3i, 1+i$, and $-1+i\sqrt{3}$ in the form $re^{i\theta}$.

$4 = 4e^{i0}$ $\quad \theta = 0 \quad r = 4$

$-2 = 2e^{i\pi}$ $\quad \theta = \pi \quad r = 2$

$3i = 3e^{i\pi/2}$ $\quad \theta = \pi/2 \quad r = 3$

$1+i = \sqrt{2}e^{i\pi/4}$ $\quad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} \quad r = \sqrt{1^2+1^2} = \sqrt{2}$

~~$-1+i\sqrt{3} = 2e^{\pm\pi/3}$ $\quad \theta = \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) = \pm\frac{\pi}{3} \quad r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$~~

$-1+i\sqrt{3} = 2e^{2\pi/3}$ $\quad \theta = \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) = \frac{2\pi}{3} \quad r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$

c) The two square roots of $re^{i\theta}$ are $\pm\sqrt{r}\,e^{i\theta/2}$. Find the square roots of the numbers $2-2i\sqrt{3}$ and $-2+2i\sqrt{3}$

$2-2i\sqrt{3} = 4e^{-\pi/3}$ $\qquad \theta = \tan^{-1}\left(-\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

$r = \sqrt{2^2+(-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

So, $\sqrt{2-2i\sqrt{3}} = \pm 2e^{-\pi/6} = \{2e^{-\pi/6}, 2e^{5\pi/6}\}$

$-2+2i\sqrt{3} = 4e^{2\pi/3}$

So, $\sqrt{-2+2i\sqrt{3}} = \pm 2e^{\pi/3} = \{2e^{\pi/3}, 2e^{-2\pi/3}\}$

3.4.1 ~~Determine the motion~~

Determine the period and frequency of the simple harmonic motion of a 4-kg mass on the end of a spring with spring constant $k = 16\,N/m$.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16\,N/m}{4\,kg}} = \boxed{2/s} = \text{angular frequency}$$

~~$f = 2\pi\omega = \boxed{4\pi/s}$ = frequency~~

~~$T = 1/f = \boxed{\frac{1}{4\pi}\,s}$ = period~~

$$f = \frac{\omega}{2\pi} = \boxed{\frac{1}{\pi}/s} = \text{frequency (cycles per second)}$$

$$T = 1/f = \boxed{\pi\,s} = \text{period}$$

s = seconds.

3.4.5. Assume the DE for the pendulum of length L is $L\theta'' + g\theta = 0$, where $g = GM/R^2$ is the gravitational acceleration at the location of the pendulum.

Two pendulums are of lengths L_1 and L_2 and – when located at the respective distances R_1 and R_2 from the Earth's center – have periods p_1 and p_2. Show that

$$\frac{p_1}{p_2} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}$$

We have simple harmonic motion with angular frequency $\omega = \sqrt{g/L}$, and so periods $T = 2\pi/\omega$.

$$p_1 = 2\pi / \sqrt{g/L_1} = 2\pi \Big/ \sqrt{\frac{GM}{R_1^2 L_1}}$$

Similarly,

$$p_2 = 2\pi \Big/ \sqrt{\frac{GM}{R_2^2 L_2}}$$

So,

$$\frac{p_1}{p_2} = \frac{2\pi \Big/ \sqrt{\frac{GM}{R_1^2 L_1}}}{2\pi \Big/ \sqrt{\frac{GM}{R_2^2 L_2}}} = \boxed{\frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}} \checkmark$$

4.18. Solve the ODE, with both damping and the damping term removed, and graph the two solutions

$m=2 \quad c=12 \quad k=50 \quad x_0=0 \quad v_0=-8$

$$(12)^2-4(2)(50) = 144-400 = -256 < 0$$

So, underdamped.

Roots of

$$2r^2+12r+50=0 \quad \text{are}$$

$$r = \frac{-12 \pm \sqrt{-256}}{2(2)} = -3 \pm 4i$$

So, the solution is:

$$x(t) = e^{-3t}\left(c_1 \cos(4t) + c_2 \sin(4t)\right)$$

$$x(0) = c_1 = 0$$

$$x'(t) = e^{-3t}\left(-4c_1 \sin(4t) + 4c_2 \cos(4t)\right) - 3e^{-3t}\left(c_1 \cos(4t) + c_2 \sin(4t)\right)$$

with $c_1 = 0$

$$x'(t) = 4e^{-3t} c_2 \cos(4t) - 3e^{-3t} c_2 \sin(4t)$$

$$= c_2 e^{-3t}\left(4\cos(4t) - 3\sin(4t)\right)$$

$$x'(0) = 4c_2 = -8 \Rightarrow c_2 = -2$$

So,

$$x'(t) = e^{-3t}\left(6\sin(4t) - 8\cos(4t)\right)$$

We can rewrite this as:

$$\cancel{x(t) = 10e^{-3t}\left(\tfrac{3}{5}\sin(4t) - \tfrac{4}{5}\cos(4t)\right)}$$

$$\cancel{\alpha = \tan^{-1}\left(\tfrac{3}{4}\right) =}$$

~~which gives us~~

$$x\,\cancel{(t) = 10e^{-3t}\cos(4t -}$$

$$x(t) = -2e^{-3t}\sin(4t)$$
$$= -2e^{-3t}\cos(4t \cancel{+} - \tfrac{\pi}{2})$$

Now, without damping we would get

$$\omega = \sqrt{\frac{50}{2}} = 5$$

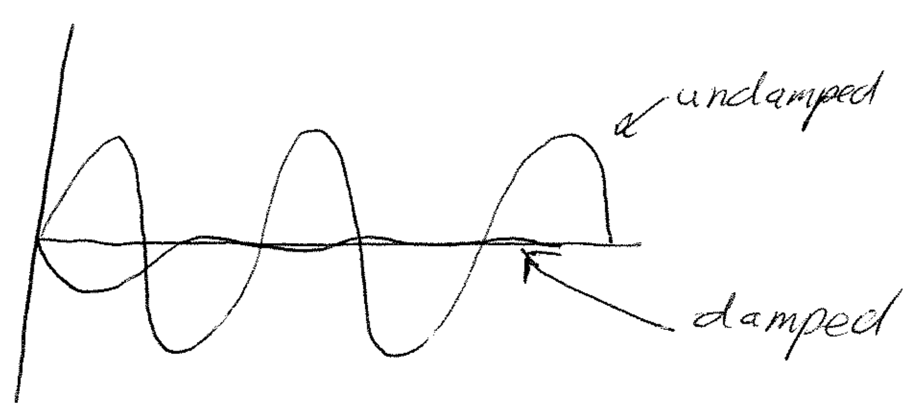
$$x(t) = c_1\cos(5t) + c_2\sin(5t)$$

$$c_1 = 0$$

$$x'(t) = 5c_2\cos(5t) \qquad c_2 = -\tfrac{8}{5}$$

$$x(t) = -\tfrac{8}{5}\sin(5t) = -\tfrac{8}{5}\cos\left(5t - \tfrac{\pi}{2}\right)$$

<u>Graph</u>



3.4.21 $m = 1 \quad c = 10 \quad k = 125 \quad x_0 = 6 \quad v_0 = 90$

$$1x'' + 10x' + 125x = 0$$

Solve characteristic equation:

$$r^2 + 10r + 125 = 0$$

$$r = \frac{-10 \pm \sqrt{10^2 - 4(1)(125)}}{2}$$

$$= \frac{-10 \pm \sqrt{100 - 500}}{2}$$

$$= -5 \pm 10i$$

So, our solution is:

$$x(t) = e^{-5t}\left(c_1 \cos(10t) + c_2 \sin(10t)\right)$$

$$x'(t) = e^{-5t}\left(-\cancel{1}0c_1 \sin(10t) + 10c_2 \cos(10t)\right) - 5e^{-5t}\left(c_1 \cos(10t) + c_2 \sin(10t)\right)$$

$$x(0) = c_1 = 6$$

$$x'(0) = 10c_2 - 30 = 50 \Rightarrow c_2 = 8$$

therefore

$$x(t) = e^{-5t}\left(6\cos(10t) + 8\sin(10t)\right)$$

We can rewrite this as:

$$x(t) = 10e^{-5t}\left(\tfrac{3}{5}\cos(10t) + \tfrac{4}{5}\sin(10t)\right)$$

$$\cancel{\tan^{-1}(\alpha) =} \quad \boxed{\alpha = \tan^{-1}\left(\tfrac{4}{3}\right)}$$

$$\boxed{x(t) = 10e^{-5t}\cos(10t - \alpha)}$$

Undamped Case:

$$\omega = \sqrt{\frac{125}{1}} = 5\sqrt{5}$$

$$x(t) = c_1\cos(5\sqrt{5}\,t) + c_2\sin(5\sqrt{5}\,t)$$

$$x(0) = 6 = c_1$$

$$x'(0) = \cancel{30}\cdot 5\sqrt{5}\,c_2 = 50 \Rightarrow c_2 = 2\sqrt{5}$$

So,

$$x(t) = 6\cos(5\sqrt{5}\,t) + 2\sqrt{5}\sin(5\sqrt{5}\,t)$$

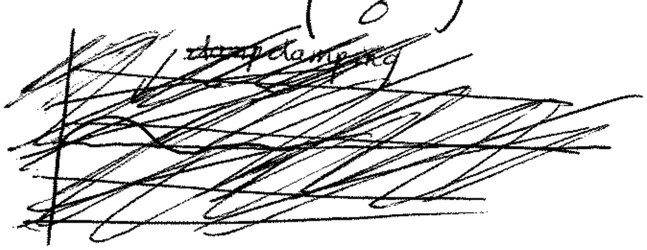
$$\sqrt{6^2 + (2\sqrt{5})^2} = \sqrt{36+20} = \sqrt{56} = 2\sqrt{14}$$

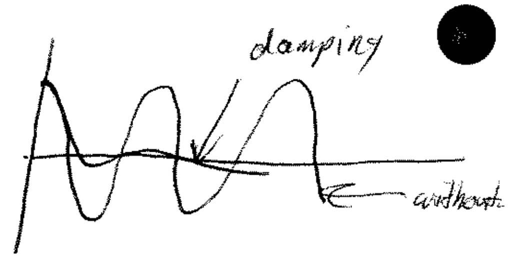
$$x(t) = 2\sqrt{14}\left(\frac{6}{2\sqrt{14}}\cos(5\sqrt{5}\,t) + \frac{2\sqrt{5}}{2\sqrt{14}}\sin(5\sqrt{5}\,t)\right)$$

$$= 2\sqrt{14}\cos(5\sqrt{5}\,t - \alpha) \qquad \cancel{\alpha = \tan^{-1}\left(\frac{6}{2\sqrt{5}}\right)}$$

$$\alpha = \tan^{-1}\left(\frac{2\sqrt{5}}{6}\right)$$

Graph





3.5.1 Find a particular solution y_p to the ODE:

$$y'' + 16y = e^{3x}$$

$$y_p = Ae^{3x} \quad y_p'' = 9Ae^{3x}$$

$$\Rightarrow \cancel{9A+25} \quad (9A + 16A)e^{3x} = e^{3x}$$

$$\Rightarrow A = \frac{1}{25}$$

So,

$$\boxed{y_p = \frac{1}{25}e^{3x}}$$

3.5.11 $y^{(3)} + 4y' = 3x - 1$

$$y_p = Ax^2 + Bx + C$$
$$y_p' = 2Ax + B$$
$$y_p'' = 2A$$
$$y_p^{(3)} = 0$$

$$\Rightarrow \cancel{2A} \quad 0 + 4(2Ax + B) = 3x - 1$$

$8A = 3 \Rightarrow A = 3/8$ C could be anything

$4B = -1 \Rightarrow B = -1/4$ Take $C = 0$

$$\boxed{y_p = \frac{3}{8}x^2 - \frac{1}{4}x}$$

3.5.23 Determine the appropritate form of y_p, but not its coefficients.

$$y'' + 4y = 3x\cos(2x)$$

$$r^2 + 4 = 0 \quad \Rightarrow r = \pm 2i$$

So, homogenous solution is:

$$y_h = c_1\cos(2x) + c_2\sin(2x)$$

first guess for y_p is:

$$y_p = (Ax+B)\cos(2x) + (Cx+D)\sin(2x)$$

but, $B\cos(2x) + D\sin(2x)$ is not LI from y_h. So we use:

$$\boxed{y_p = (Ax^2+Bx)\cos(2x) + (Cx^2+Dx)\sin(2x)}$$

3.5.28 $y^{(4)} + 9y'' = (x^2+1)\sin 3x$

characteristic equation:

$$r^4 + 9r^2 = r^2(r+3i)(r-3i)$$

has roots $r = 0, 3i, -3i$ with 0 repeated.

So,

$$y_h = c_1 + c_2x + c_3\sin(3x) + c_4\cos(3x)$$

Our first guess for y_p would be:

$$y_p = (Ax^2+Bx+C)\sin(3x) + (Dx^2+Ex+F)\cos(3x)$$

but, that's not LI from y_h (the $C\sin(3x) + F\cos(3x)$ term)

So,

$$y_p = (Ax^3+Bx^2+Cx)\sin(3x) + (Dx^3+Ex^2+Fx)\cos(3x)$$

3.5.35 Solve the initial value problem

$$y''-2y'+2y = x+1 \qquad y(0)=3, \quad y'(0)=0$$

Solve the characteristic equation:

$$r^2-2r+2=0 = (r-1)^2+1=0$$

$$\Rightarrow r = 1 \pm i$$

So, we get:

$$y_h = e^x(c_1\cos(x) + c_2\sin(x))$$

Now, the guess for y_p would be:

$$y_p = Ax+B$$

which has each term LI from each term of y_h.

So,

$$y_p = Ax+B$$
$$y_p' = A$$
$$y_p'' = 0$$

Plugging in we get:

$$-2A + 2(Ax+B) = x+1$$

$$\Rightarrow 2A=1 \qquad A=\tfrac{1}{2}$$

$$-2A+2B=1 \qquad B=1$$

and so:

$$y(x) = y_h + y_p$$

$$= e^x(c_1 \cos(x) + c_2 \sin(x)) + \frac{1}{2}x + 1$$

Using our initial conditions

$$y(0) = c_1 + 1 = 3 \Rightarrow c_1 = 2$$

$$y'(x) = e^x(2\cos(x) + c_2 \sin(x)) + e^x(-2\sin(x) + c_2\cos(x)) + \frac{1}{2}$$

$$y'(0) = 2 + c_2 + \frac{1}{2} = 0 \Rightarrow c_2 = -5/2$$

So,

$$\boxed{y(x) = e^x(2\cos(x) - \frac{5}{2}\sin(x)) + \frac{1}{2}x + 1}$$

3. 5.47 Use the method of variation of parameters to solve the ODE:

$$y'' + 3y' + 2y = 4e^x$$

Solve for y_h

$$r^2 + 3r + 2 = (r+1)(r+2) \qquad r = -1, -2$$

So,

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$$

Now,
$$y_1 = e^{-x} \qquad y_2 = e^{-2x}$$

$$W(x) = W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$\int \frac{y_2(x) f(x)}{W(x)} = \int \frac{e^{-2x}(4e^{x})}{-e^{-3x}}$$

$$= -\int 4e^{2x} = -2e^{2x}$$

$$\int \frac{y_1(x) f(x)}{W(x)} = \int \frac{e^{-x}(4e^{x})}{-e^{-3x}}$$

$$= -\int 4e^{3x} = -\frac{4}{3} e^{3x}$$

So,

$$y_p = -e^{-x}(-2e^{2x}) + e^{-2x}\left(-\frac{4}{3}e^{3x}\right)$$

$$= 2e^{x} - \tfrac{4}{3}e^{x} = \boxed{\tfrac{2}{3}e^{x}}$$

Check:
$$y_p'' = \tfrac{2}{3}e^{x} \quad y_p' = \tfrac{2}{3}e^{x} \quad y_p = \tfrac{2}{3}e^{x}$$

$$y_p'' + 3y_p' + 2y_p = \left(\tfrac{2}{3} + 2 + \tfrac{4}{3}\right)e^{x} = \tfrac{12}{3}e^{x} = 4e^{x}$$ ✓

3.5.56 $y'' - 4y = xe^x$

Solving for y_h

$\cancel{y_h}$ $r^2 - 4 = 0$ $r = \pm 2$

$y_1 = e^{2x}$ $y_2 = e^{-2x}$

$\cancel{\int}$ $W(x) = W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}$

$= -2 - 2 = -4$

$\int \frac{y_2 f}{W} = \int \frac{e^{-2x} x e^x}{-4} = -\frac{1}{4} \int x e^{-x}$

$= \frac{1}{4} x e^{-x} + \frac{1}{4} e^{-x}$

$\int \frac{y_1 f}{W} = \int \frac{e^{2x} x e^x}{-4} = -\frac{1}{4} \int x e^{3x}$

$= -\frac{1}{12} x e^{3x} + \frac{1}{36} e^{3x}$

So,

$y_p = -e^{2x}\left(\frac{1}{4} x e^{-x} + \frac{1}{4} e^{-x}\right) + e^{-2x}\left(-\frac{1}{12} x e^{3x} + \frac{1}{36} e^{3x}\right)$

$= -\frac{1}{4} x e^x - \frac{1}{4} e^x - \frac{1}{12} x e^x + \frac{1}{36} e^x$

$= \boxed{-\frac{1}{3} x e^x - \frac{2}{9} e^x}$