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Math 2280

Assignment #4 Solutions

2.4.1 Apply Euler's method twice to approximate the solution on the interval $[0, \frac{1}{2}]$, first with step size .25 then with step size .1. Compare the 3 decimal place solutions of the approximations with the actual value at $x = \frac{1}{2}$.

$$y' = -y \qquad y(0) = 2 \qquad y(x) = 2e^{-x}$$

For $h = .25$

$$y_0 = 2 \qquad x = 0$$

$$y_1 = y_0 + (.25)(-2) \qquad x = .25$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

$$y_2 = \frac{3}{2} + (.25)\left(-\frac{3}{2}\right) \qquad x = .5$$

$$= \frac{3}{2} - \frac{3}{8} = \boxed{\frac{9}{8}}$$

For $h = .1$

$$y_0 = 2$$

$$y_1 = 2 + (.1)(-2) \qquad x = .1$$

$$= 2 - .2 = 1.8$$

$$y_2 = 1.8 + (.1)(-1.8) \qquad x = .2$$

$$= 1.8 - .18 = 1.62$$

$y_3 = 1.62 + (.1)(-1.62) \qquad x = .3$

$= 1.458$

$y_4 = 1.458 + (.1)(-1.458) \qquad x = .4$

$= 1.3122$

$y_5 = 1.3122 + (.1)(-1.3122) \qquad x = .5$

$= 1.18098$

$$\begin{array}{r} 31220 \\ -13122 \\ \hline 18098 \end{array}$$

Now, the exact value of y at $x = .5$ is:

$y(.5) = 2e^{-.5} = 1.213$

So, for $h = .25$ off by:

$9/8 = 1.125 \qquad$ difference: $1.213 - 1.125 = \boxed{.088}$

While for $h = .1$ off by:

$1.213 - 1.181 = \boxed{.032}$

2.4.5. $y' = y - x - 1 \qquad y(0) = 1 \qquad y(x) = 2 + x - e^x$

$h = .25$

$y_0 = 1 \qquad x = 0$

$y_1 = 1 + (.25)(1 - 0 - 1) \qquad x = .25$

$= 1$

$y_2 = 1 + (.25)(1 - .25 - 1)$

$= 1 - \frac{1}{16} = \boxed{\frac{15}{16}}$

$h = .1$

$y_0 = 1 \qquad x = 0$

$y_1 = 1 + (.10)(1 - 0 - 1) \qquad x = .1$

$= 1$

$y_2 = 1 + (.1)(1 - .1 - 1) \qquad x = .2$

$= \cancel{1.01}\ .99$

$y_3 = \cancel{1.01 + (.1)(1.01 - .2 - 1) \quad x = .3}$

$.99 + (.1)(.99 - .2 - 1)$

$= \cancel{.991} = .969$

$y_4 = \cancel{.991 + (.1)(.991 - .3 - 1)} \qquad x = .4$

$.969 + (.1)(.969 - .3 - 1)$

$= \cancel{.9601} = .1359$

$y_5 = \cancel{.9601 + .1(.9601-.4-1)}$ $\quad x = .9$

$.9359 + .1(.9359 - .4 - 1)$

$= \cancel{.91611}\ .88949$

Actual value at $x = .9$

$y(.9) = 2 + .9 - e^{.9} = .891$

So, $h = .25$ off by:

$.93\cancel{8} - .851 = .087$

$h = .1$ off by:

$.889 - .851 = .038$

~~.916 - .891 = .065~~

Not a huge diference, but $h = .1$ is the better approximation

4.9. $y' = \frac{1}{4}(1+y^2)$, $y(0) = 1$ $\quad y(x) = \tan(\frac{1}{4}(x+\pi))$

$y_0 = 1$ $\quad x = 0$

$y_1 = 1 + \frac{1}{4}(\frac{1}{4}(1+1^2))$ $\quad x = .25$

$= 1 + \frac{1}{8} = \frac{9}{8}$

$y_2 = \frac{9}{8} + \frac{1}{4}(\frac{1}{4}(1+(\frac{9}{8})^2))$ $\quad \cancel{x = .25}\ x = .5$

$= 1.27$

$h = .1$

$y_0 = \cancel{0}1$ $\qquad x = 0$

$y_1 = 1 + (.1)(\frac{1}{4}(1+1^2))$ $\qquad x = .1$

$= 1 + .05 = 1.05$

$y_2 = 1.05 + (.1)(\frac{1}{4}(1+(1.05)^2))$ $\qquad x = .2$

$= \cancel{1.26}\ 1.10$

$y_3 = 1.10 + (.1)(\frac{1}{4}(1+(1.10)^2))$ $\qquad x = .3$

$= 1.158$

$y_4 = 1.158 + (.1)(\frac{1}{4}(1+1.158^2))$ $\qquad x = .4$

$= 1.217$

$y_5 = \cancel{1.279}\ 1.217 + (.1)(\frac{1}{4}((1+1.217^2)))$ $\qquad x = .5$

$= 1.279$

Now, the actual value is:

$y(.5) = \tan(\frac{1}{4}(.5+\pi)) = 1.287$

So,

$h = .25$ off by:

$1.287 - 1.27 = .017.$

$h = .1$ off by:

$1.287 - 1.279 = .008.$

4.26. Suppose the deer population $P(t)$ in a small forest initially numbers 25 and satisfies the logistic equation

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2$$

(with t in months). Use Euler's method with a programmable calculator or computer to approximate the solution for 10 years, first with step size $h=1$ and then with $h=.5$, rounding off approximate P-values to integral number of deer. What percentage of the limiting population of 75 deer has been obtained after 5 years? After 10 years?

Note: If we round every step, the population doesn't change.

$h = 1$ month			$h = .5$ month	
t	P	Rounded P	P	Rounded P
0	25	25	25	25
1 month	25.375	25	25.376	25
2 months	25.753	26	25.754	26
1 year	29.667	30	29.675	30
5 years	49.389	49	49.390	49
10 years	66.180	66	66.235	66

Percentage after 5 years: 65%
Percentage after 10 years: 88%

Note: $h=1$ and $h=.5$ almost identical!

2.4.30 Apply Euler's method with successively smaller step sizes on the interval $[0, 2]$ to verify emperically that the solution of the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 0$$

has a vertical asymptote near $x = 2.003147$.

x-value	y-values			
	$h = .5$	$h = .1$	$h = .01$	$h = .001$
.5	0	.030	.041	.042
1	.125	.401	.344	.350
1.5	.633	1.213	1.479	1.518
2	1.958	5.852	28.393	142.627

We see as h gets small as we approach x=2 we get very large y values. This is caused by the asymptote.

Note: I had to write a Java program to do this.

3.1.1. Verify y_1 and y_2 are solutions, and then find constants $c_{1,2}$ such that $y = c_1 y_1 + c_2 y_2$ satisfies the given initial conditions

$$y'' - y = 0 \qquad y_1 = e^x \quad y_2 = e^{-x} \quad \begin{matrix} y(0) = 0 \\ y'(0) = 5 \end{matrix}$$

$$y_1' = e^x$$
$$y_1'' = e^x$$

$$y_1'' - y_1 = e^x - e^x = 0$$

$$y_2' = -e^{-x}$$
$$y_2'' = e^{-x}$$

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0$$

So, both y_1 and y_2 work.

$$y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{-x}$$
$$y' = c_1 e^x - c_2 e^{-x}$$

So,
$$y(0) = c_1 + c_2 = 0$$
$$y'(0) = c_1 - c_2 = 5$$
$$\Rightarrow c_1 = \tfrac{5}{2} \quad c_2 = -\tfrac{5}{2}$$

So,
$$\boxed{y(x) = \tfrac{1}{2}(5e^x - 5e^{-x})}$$

3.1.16. $x^2y'' + xy' + y = 0$ $\quad y_1 = \cos(\ln x)$

$y_2 = \sin(\ln x)$

$y(1) = 2 \qquad y'(1) = 3$

$$y_1' = \frac{-\sin(\ln x)}{x} \qquad y_1'' = \frac{-\cos(\ln x) + \sin(\ln x)}{x^2}$$

$$x^2y_1'' + xy_1' + y_1 = -\cos(\ln x) + \sin(\ln x) - \sin(\ln x) + \cos(\ln x) = 0$$

So, checks out for y_1.

$$y_2' = \frac{\cos(\ln x)}{x} \qquad y_2'' = \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}$$

$$x^2y_2'' + xy_2' + y_2 = -\sin(\ln x) - \cos(\ln x) + \cos(\ln x) + \sin(\ln x) = 0$$

So, checks out for y_2.

$$y = c_1y_1 + c_2y_2 = c_1\cos(\ln x) + c_2\sin(\ln x)$$

$$y(1) = c_1\cos(\ln(1)) + c_2\sin(\ln(1)) = c_1 = 2$$

$$y'(x) = \frac{-2\sin(\ln x)}{x} + \frac{c_2\cos(\ln x)}{x}$$

$$y'(1) = \frac{-2\sin(\ln 1)}{1} + \frac{c_2\cos(\ln 1)}{1} = c_2 = 3$$

$$\Rightarrow \boxed{y(x) = 2\cos(\ln(x)) + 3\sin(\ln(x))}$$

3.1.18 Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that if $c^2 \neq 1$, then $y = cx^3$ is not a solution.

Suppose $y = cx^3$ then $y' = 3cx^2$ and $y'' = 6cx$
So,

$$yy'' = 6c^2x^4 = 6x^4$$

if and only if $c^2 = 1$, and so $c = \pm 1$.

So,

$$y = x^3 \quad \text{or} \quad y = -x^3$$

work, but no other c works.

3-1.24. Determine if $f(x)$ and $g(x)$ are linearly independent on the real line $\mathbb{R}$.

$f(x) = \sin^2 x \qquad g(x) = 1 - \cos 2x$

Now, $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ (basic trig. identity)

So, $2g(x) = f(x)$

and they are not linearly independent

Other proof:

$$W(f,g) = \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cos x & 2\sin 2x \end{vmatrix} = 2\sin^2 x \sin 2x - 2\sin x \cos x (1 - \cos 2x) = 0 \text{ when } x = 0.$$

3.1.39 Find a general solution to the ODE:

$$4y'' + 4y' + y = 0$$

Characteristic Equation:

$$4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2$$

So, $r = -\frac{1}{2}$ is a root of order 2.

So, the general solution will be:

$$\boxed{y(x) = c_1 e^{-x/2} + c_2 x e^{-x/2}}$$

.2.1 Show that the following functions are linearly dependent on $\mathbb{R}$.

$f(x) = 2x, \quad g(x) = 3x^2, \quad h(x) = 5x - 8x^2$

$$6h(x) + 16g(x) - 15f(x)$$

$$= 6(5x - 8x^2) + 16(3x^2) - 15(2x)$$
$$= 30x - 48x^2 + 48x^2 - 30x$$
$$= \boxed{0}$$

3.2.10 Use the Wronskian to prove the following functions are linearly independent

$$f(x) = e^x \quad g(x) = x^{-2} \quad h(x) = x^{-2}\ln x \quad x > 0$$

$$W(f,g,h) = \begin{vmatrix} e^x & x^{-2} & x^{-2}\ln x \\ e^x & -2x^{-3} & x^{-3} - 2x^{-3}\ln x \\ e^x & 6x^{-4} & -3x^{-4} - 2x^{-4} + 6x^{-4}\ln x \end{vmatrix}$$

$$\cancel{= e^x \begin{vmatrix} -\frac{2}{x^2} & \frac{1}{x^3} - \frac{2\ln y}{x^3} \\ \frac{6}{x^4} & -5 + 6\ln y \end{vmatrix}}$$

Evaluate at $x = 1$ to get

$$W(f,g,h)(1) = \begin{vmatrix} e & 1 & 0 \\ e & -2 & 1 \\ e & 6 & -5 \end{vmatrix}$$

$$= e(10-6) - e(-5-0) + e(1-0)$$

$$= 4e + 5e + e = 10e \neq 0$$

So, linearly independent

3.2-16 Find a particular solution to the given ODE with given linearly independent homogenous solutions that satisfy the given initial conditions

$$y^{(3)} - 5y'' + 8y' - 4y = 0; \quad y(0) = 1 \quad y'(0) = 4 \quad y''(0) = 0;$$

$$y_1 = e^x \quad y_2 = e^{2x} \quad y_3 = xe^{2x}$$

$$y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3$$
$$= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$$

$$y(0) = c_1 + c_2 \cancel{+ c_3}$$

$$y'(x) = c_1 e^x + 2c_2 e^{2x} + 2c_3 x e^{2x} + c_3 e^{2x}$$

$$y'(0) = c_1 + 2c_2 + c_3$$

$$y''(x) = c_1 e^x + 4c_2 e^{2x} + 4c_3 x e^{2x} + + 4c_3 e^{2x}$$

$$y''(0) = c_1 + 4c_2 + 4c_3$$

$$\Rightarrow \begin{aligned} c_1 + c_2 &= 1 \\ c_1 + 2c_2 + c_3 &= 4 \\ c_1 + 4c_2 + 4c_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_2 + c_3 &= 3 \\ 3c_2 + 4c_3 &= -1 \end{aligned} \qquad c_2 = 3 - c_3$$

$$3(3 - c_3) + 4c_3 = -1 \Rightarrow 9 + c_3 = -1 \Rightarrow c_3 = -10$$
$$\Rightarrow c_2 = 13 \qquad c_1 = -12$$

So,

$$\boxed{y(x) = -12e^x + 13e^{2x} - 10xe^{2x}}$$

3-2-24 Find a solution to the given ODE that satisfies the initial conditions with the given y_c and y_p

$$y''-2y'+2y=2x \qquad y(0)=4 \qquad y'(0)=8$$

$$y_c=c_1e^x\cos x+c_2e^x\sin x \qquad y_p=x+1$$

$$y=c_1e^x\cos x+c_2e^x\sin x+x+1$$

$$y'=-c_1e^x\sin x+c_1e^x\cos x+c_2e^x\cos x+c_2e^x\sin x+1$$
$$=(-c_1+c_2)e^x\sin x+(c_1+c_2)e^x\cos x+1$$

$$y(0)=c_1+1=4 \Rightarrow c_1=3$$
$$y'(0)=c_1+c_2+1=8 \Rightarrow 3+c_2+1=8 \Rightarrow c_2=4$$

So,

$$\boxed{y(x)=3e^x\cos x+4e^x\sin x+x+1}$$

3.2.31 This problem indicates why we can only impose n initial conditions on the solution of an n-th order linear differential equation.

a) Given the equation

$$y''+py'+qy=0$$

explain why the value of $y''(a)$ is determined by the values of $y(a)$ and $y'(a)$.

Well,

$$y''(a) = -p(a)y'(a) - q(a)y(a).$$

So, if $y(a)$, $p(a)$, $y'(a)$, and $q(a)$ are known then $y''(a)$ is determined.

b) Prove that the equation

$$y''-2y'-5y=0$$

has a solution satisfying the conditions

$$y(0)=1 \quad y'(0)=0 \quad \text{and} \quad y''(0)=C$$

if and only if $C=5$.

Well,

$$y''(0) = 2y'(0)+5y(0) = 2(0)+5(1) = 5$$

So, we must have $y''(0)=5$.