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Math 2280
Spring 2008

Assignment #2 Solutions

1.4

Find general solutions to the following O.D.Es.

1. $\frac{dy}{dx} + 2xy = 0$

$\Rightarrow \frac{dy}{dx} = -2xy \Rightarrow \frac{dy}{y} = -2x\,dx$

$\Rightarrow \ln|y| = -x^2 + C$

$\Rightarrow \boxed{y = Ce^{-x^2}}$

3. $\frac{dy}{dx} = y\sin x$

$\Rightarrow \frac{dy}{y} = \sin x\,dx$

$\Rightarrow \ln|y| = -\cos x + C$

$\Rightarrow \boxed{y = Ce^{-\cos x}}$

17.

$$\frac{dy}{dx} = 1+x+y+xy$$

$$\Rightarrow \frac{dy}{dx} = (1+x)(1+y)$$

$$\Rightarrow \frac{dy}{1+y} = (1+x)\,dy$$

$$\Rightarrow \ln|1+y| = \frac{x^2}{2} + x + C$$

$$\Rightarrow y+1 = Ce^{\frac{x^2}{2}+x}$$

$$\Rightarrow \boxed{y = Ce^{x^2/2+x} - 1}$$

<u>Find explicit particular solutions</u>

19. $\frac{dy}{dx} = ye^x \qquad y(0) = 2e$

$$\frac{dy}{y} = e^x dy$$

$$\ln|y| = e^x + C$$

$$\Rightarrow y(x) = Ce^{e^x}$$

$$y(0) = Ce^{e^0} = Ce = 2e \Rightarrow C = 2.$$

So,

$$\boxed{y(x) = 2e^{e^x}}$$

31. Discuss the difference between the differential equations:

$$\left(\frac{dy}{dx}\right)^2 = 4y \quad \text{and} \quad \frac{dy}{dx} = 2\sqrt{y}.$$

Do they have the same solution curves? Why or why not? Determine the points (a,b) on the plane where the initial value problem $y' = 2\sqrt{y}$, $y(a) = b$ has a) no solution, b) a unique solution, c) infinitely many solutions

A solution to $\left(\frac{dy}{dx}\right)^2 = 4y$ could solve either $\frac{dy}{dx} = 2\sqrt{y}$ or $\frac{dy}{dx} = -2\sqrt{y}$. Any differential "patch up" of these solutions would work, so the solution curves are different. (There are more for $\left(\frac{dy}{dx}\right)^2 = 4y$).

a) $\frac{dy}{dx} = 2\sqrt{y}$ $\qquad D_y(2\sqrt{y}) = \frac{1}{\sqrt{y}}$

Not continuous for $y=0$, does not exist for $y<0$. So.

No solution if $b<0$.

~~In f.~~
A unique solution if ~~x~~ $b>0$, as both $f(x,y)$ and $D_y(f(x,y))$ are continuous there.

Two ~~Infinitely many~~ solutions if $b=0$. If $(a,b)=(a,0)$ the solving the ODE

$$\frac{dy}{dx} = 2\sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}} = 2dx$$

$$\Rightarrow 2\sqrt{y} = 2x + C \Rightarrow y(x) = (\cancel{2}x+C)^2$$

If $y(a)=0$ then $C=-\cancel{2}a$.

$$\cancel{y(x)=(2x-2a)^2} \quad y(x)=(x-a)^2$$

But, $y(x)=0$ also works.

There seem to be no points with infinitely many solutions. For $y_0=0$, there are 2 possible solutions

1.4.35

Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

$$\frac{1}{6} = \left(\frac{1}{2}\right)^{(t/T_{1/2})}$$

$$\Rightarrow \quad \log 6 = (t/T_{1/2}) \log 2$$

$$\Rightarrow \quad t = T_{1/2} \left(\frac{\log 6}{\log 2}\right) = 5700\,\text{years} \left(\frac{\log 6}{\log 2}\right)$$

$$= 14{,}734 \text{ years}$$

$$\approx \boxed{15{,}000 \text{ years}}$$

1.4.53

Thousands of years ago ancestors of the Native Americans crossed the Bering Strait from Asia and entered the western hemisphere. Since then, they have fanned out across North and South America. The single language that the original Native Americans spoke has split into many Indian "language families." Assume that the number of these families has been multiplied by 1.5 every 6000 years. There are now 150 Native American language families in the western hemisphere. About when did the ancestors of today's native Americans arrive?

$$150 = (1.5)^{(t/6000)}$$

$$\Rightarrow \log(150) = \frac{t}{6000} \log(1.5)$$

$$\Rightarrow t = 6000 \left(\frac{\log(150)}{\log(1.5)} \right.$$

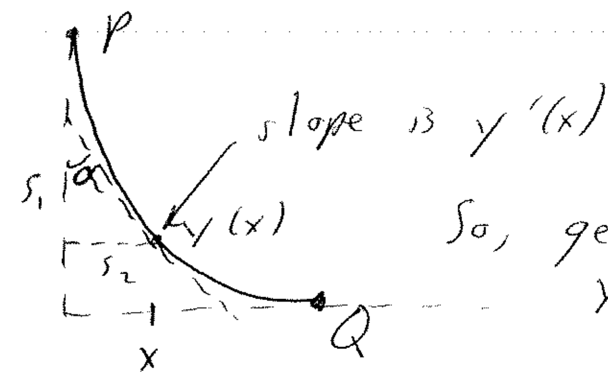
$$\approx \boxed{74{,}000 \text{ years ago}}$$

1.4.68

The brachistochrone problem asks what shape the wire should be in order to minimize the bead's time of descent from P to Q. In June of 1696, John Bernoulli proposed this problem as a public challenge, with a 6-month deadline (later extended to Easter 1697 at George Leibniz's request). Isaac Newton, then retired from academic life and serving as Warden of the Mint in London, received Bernoulli's challenge on January 29, 1697. The very next day he communicated his own solution - the curve of minimal descent time is an arc of an inverted cycloid - to the Royal Society of London. For a modern derivation of this result, suppose the bead starts from rest at the origin P and let $y = y(x)$ be the equation of the desired curve in a a coordinate system with the y-axis pointing downward. Then a mechanical analogue of Snell's law in optics implies that

$$\frac{\sin\alpha}{v} = \text{constant}$$

where α denotes the angle of deflection (from the vertical) of the tangent line to the curve - so $\cot\alpha = y'(x)$ (why?) and $v = \sqrt{2gy}$ is the bead's velocity when it has descended a distance y vertically (from $KE = \frac{1}{2}mv^2 = mgy = -PE$).



So, geometrically we see $y'(x) = s_1/s_2 = \cot(\alpha)$

a) Derive from $\dfrac{\sin\alpha}{v}$ = constant the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$$

where a is ~~the~~ an appropriate constant.

Call the constant in $\dfrac{\sin\alpha}{v}$ = constant C.

Then, as

$$y'(x) = \cot(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{\cos(\alpha)}{vC}$$

and $\cos(\alpha) = \sqrt{1-\sin^2(\alpha)} = \sqrt{1-v^2C^2}$

we have

$$\frac{dy}{dx} = \frac{\sqrt{1-v^2C^2}}{vC}$$

Now, $v = \sqrt{2gy}$ so this equals

$$\frac{dy}{dx} = \frac{\sqrt{1-2C^2gy}}{C\sqrt{2gy}} = \frac{\sqrt{1-2C^2gy}}{\sqrt{2C^2gy}} = \sqrt{\frac{2a-y}{y}}$$

for $a = \dfrac{1}{4C^2g}$

b) Substitute $y = 2a\sin^2(t)$; $dy = 4a\sin(t)\cos(t)dt$ in ii) to derive the solution

$$x = a(2t - \sin(2t)) \qquad y = a(1-\cos(2t))$$

for which $t = y = 0$ when $x = 0$. Finally, the substitution $\theta = 2t$ yields the standard parametric equation of a cycloid.

$$y = 2a\sin^2(t) \qquad dy = 4a\sin(t)\cos(t)dt$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$$

becomes

$$\frac{4a\sin(t)\cos(t)dt}{dy} = \sqrt{\frac{2a - 2a\sin^2(t)}{2a\sin^2(t)}}$$

$$\Rightarrow 4a\sin(t)\cos(t)\left(\frac{dt}{dx}\right) = \cancel{\sqrt{2a}}\sqrt{\frac{1-\sin^2(t)}{\sin^2(t)}} = \cancel{\sqrt{2a}}\,\frac{\cos(t)}{\sin(t)} = \cancel{\sqrt{2a}}\cot(t)$$

$$\Rightarrow \frac{2\cancel{\sqrt{2a}}}{4a}\sin^2(t)\left(\frac{dt}{dx}\right) = 1$$

$$\Rightarrow \frac{2\cancel{\sqrt{2a}}}{4a}\sin^2(t) = \left(\frac{dx}{dt}\right) \qquad \sin^2(t) = \frac{1-\cos(2t)}{2}$$

$$\Rightarrow \frac{\cancel{\sqrt{2a}}}{2a}\left(1 - \cos(2t)\right) = \frac{dx}{dt}$$

$$\Rightarrow x(t) = 2a\left(t - \frac{\sin(2t)}{2}\right) + C$$
$$= a(2t - \sin(2t)) + C \qquad x(0) = 0 \Rightarrow C = 0$$

$$x(t) = a(2t - \sin(2t))$$

So,

$$x(t) = a(2t - \sin(2t))$$

$$y(t) = 2a\sin^2(t) = 2a\left(\frac{1-\cos(2t)}{2}\right)$$

$$= a(1-\cos(2t)).$$

So,

$$x(t) = a(2t - \sin(2t)) \quad y(t) = a(1-\cos(2t))$$

Q.E.D.

1.5 Find the general and particular solutions ~~#~~ to the given ODE

1. $y'+y=2$, $y(0)=0$

$\rho(x)=e^{\int dx}=e^x$

$\Rightarrow e^x y'+e^x y=2e^x$

$\Rightarrow (e^x y)'=2e^x$

$\Rightarrow e^x y=\int 2e^x=2e^x+C$

$\Rightarrow \boxed{y=Ce^{-x}+2}$

$y(0)=C+2=0 \quad \Rightarrow C=-2$

$\boxed{y=2-2e^{-x}}$

15. $y'+2xy=x$ $\quad y(0)=-2$

$\rho(x)=e^{\int 2x\,dx}=e^{x^2}$

$\Rightarrow y'e^{x^2}+2xe^{x^2}y=xe^{x^2}$

$\Rightarrow (e^{x^2}y)'=xe^{x^2}$

$\Rightarrow e^{x^2}y=\int xe^{x^2}=\frac{1}{2}e^{x^2}+C$

$\Rightarrow \boxed{y=Ce^{-x^2}+\frac{1}{2}}$

$y(0)=C+\frac{1}{2}=-2 \Rightarrow C=-\frac{5}{2}$ So, $y=\boxed{\dfrac{1-5e^{-x^2}}{2}}$

21. $xy' = 3y + x^4 \cos x \qquad y(2\pi) = 0$

$\Rightarrow y' - \frac{3}{x} y = x^3 \cos x \qquad$ (assuming $x > 0$)

$p(x) = e^{\int -\frac{3}{x} dx} = e^{-3 \ln x} = x^{-3} = \frac{1}{x^3}$

$\Rightarrow \quad \frac{y'}{x^3} - \frac{3y}{x^4} = \cos x$

$\Rightarrow \quad \left(\frac{y}{x^3}\right)' = \cos x \Rightarrow \frac{y}{x^3} = \sin x + C$

$\Rightarrow \boxed{y = Cx^3 + x^3 \sin x}$

$y(2\pi) = C(2\pi)^3 + (2\pi)^3 \sin(2\pi) = C(2\pi)^3 = 0$

$\Rightarrow C = 0$

So, $\boxed{y = x^3 \sin x}$

29. Express the general solution of $dy/dx = 1 + 2xy$ in terms of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

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The DE $\frac{dy}{dx} = 1 + 2xy$ can be written

$y' - 2xy = 1$

$$\rho(x) = e^{\int -2x\,dx} = e^{-x^2}$$

$$\Rightarrow e^{-x^2}y' - 2xe^{-x^2}y = e^{-x^2}$$

$$\Rightarrow (e^{-x^2}y)' = e^{-x^2}$$

$$\Rightarrow e^{-x^2}y = \int e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C$$

$$\Rightarrow \boxed{y = e^{x^2}\left(\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C\right)}$$

38. Consider the cascade of two tanks shown in figure 1.5.5, with $V_1 = 100$ gal and $V_2 = 200$ gal the volumes of brine in the two tanks. Each tank also initially contains 50 lb of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1.

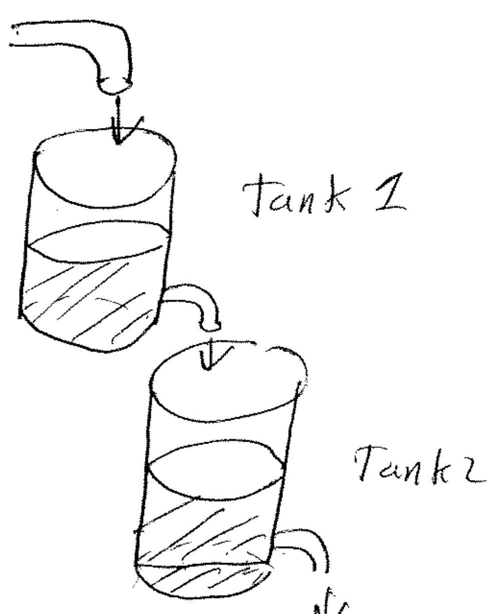

Figure 1.5.5.

a) Find the amount $x(t)$ of salt in tank 1 at time t.

If we assume instantaneow mixing we have, if x is the amount of salt, then

$$\frac{dx}{dt} = -x\left(\frac{5}{100}\right)$$

$$\Rightarrow \quad x(t) = Ce^{-\frac{5}{100}t}$$
$$x(0) = C = 50$$

$$\Rightarrow \quad \boxed{x(t) = 50e^{-\frac{5}{100}t}}$$

b) Suppose that $y(t)$ is the amount of salt in container 2 at time t. Show that it satisfies

$$\bigstar \quad \frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}$$

and then solve for $y(t)$.

It gains $5x/100$ in time Δt from tank 1 and, assuming instant mixing, loses $5y/200$ in time Δt. Taking the limit as $\Delta t \to 0$ we get $\bigstar$.

$$\Rightarrow \quad \frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}$$

$$\Rightarrow \quad y' + \frac{5}{200}y = 2.5e^{-5/100\,t}$$

$$\rho(t) = e^{+\int \frac{5}{200}dt} = e^{\frac{5}{200}t}$$

$$\Rightarrow \quad (ye^{5/200\,t})' = 2.5e^{-5/200\,t}$$

$$\Rightarrow \quad ye^{\frac{5}{200}t} = -100e^{-5/200\,t} + C$$

$$\Rightarrow \quad y(t) = Ce^{-\frac{5}{200}t} - 100e^{-5/100\,t}$$

$$y(0) = 0 = C - 100 \Rightarrow C = 100.$$

So, $$\boxed{y(t) = 100\left(e^{-\frac{5}{200}t} - e^{-5/100\,t}\right)}$$

c) ~~Maximum when~~
Find the maximum of $y(t)$.

Find where

$$y'(t) = 0$$

$$y'(t) = 5e^{-5/100\,t} - 2.5e^{-5/100\,t} = 0$$

$$\Rightarrow \quad 2 = e^{\frac{5}{100}t}$$
$$\Rightarrow \quad \boxed{40\ln 2 = t}$$

42. Suppose that a falling hailstone with density $\delta = 1$ starts from rest with a negligible radius $r=0$. Thereafter its radius increases at a rate $r = kt$ (k a constant) as it grows by accretion during its fall. Use Newton's second law according to which the net force F acting on a possibly variable mass m equals the time rate of change dp/dt of its momentum $p = mv$ — to set up ~~san~~ and solve the initial value problem

$$\frac{d}{dt}(mv) = mg \qquad v(0) = 0$$

where m is the variable mass of the hailstone, $v = dy/dt$ is its velocity, and the positive y-axis points downward. Then show that $\frac{dv}{dt} = g/4$. Thus, the hailstone falls as though it were under one-forth the influence of gravity.

$$\frac{d}{dt}(mv) = \frac{dm}{dt}v + m\frac{dv}{dt}$$

$$m(t) = \delta \frac{4}{3}\pi r(t)^3 \qquad \frac{dm}{dt} = 4\delta\pi r(t)^2 \frac{dr}{dt}$$

$$= 4\delta\pi r(t)^2 (k)$$

as

$$\frac{dr}{dt} = k$$

So, $$\frac{dm}{dt} = \frac{3mk}{r}$$

and so

$$m\frac{dv}{dt} + \frac{3mKv\cancel{t}}{r(t)} = mg$$

$$\Rightarrow \frac{dv}{dt} + \frac{3k}{kt}v = g \qquad \text{using } r(t) = kt$$

$$\Rightarrow \frac{dv}{dt} + \frac{3}{t}v = g$$

Solving this ODE

$$p(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = t^3$$

$$\Rightarrow \quad (vt^3)' = gt^3$$

$$\Rightarrow \quad vt^3 = \frac{g}{4}t^4$$

$$\Rightarrow \quad \boxed{v(t) = \frac{g}{4}t \Rightarrow \frac{dv}{dt} = \frac{g}{4}}$$

.6.4 Find the general solution:

1. $(x+y)y' = x-y$

$\Rightarrow (x+y)\frac{dy}{dx} = x-y$

$\Rightarrow (x+y)dy = (x-y)dx$

$\Rightarrow (y-x)dx + (x+y)dy = 0$

$M(x,y) = y-x \quad N(x,y) = x+y$

$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1$

So, it's exact, and we solve:

$F(x,y) = \int M(x,y)\,dx = yx - \frac{x^2}{2} + c(y)$

$\frac{\partial F}{\partial y} = x + c'(y) = x+y \Rightarrow c'(y) = y \Rightarrow c(y) = \frac{y^2}{2} + C$

So, $\boxed{F(x,y) = \frac{y^2}{2} + yx - \frac{x^2}{2} + C}$ Or,

$\boxed{\frac{y^2}{2} + yx - \frac{x^2}{2} = C}$

11. $(x^2-y^2)y' = 2xy$

$\Rightarrow (x^2-y^2)\frac{dy}{dx} = 2xy$

$\Rightarrow (x^2-y^2)dy - (2xy)dx = 0$

$M(x,y) = -2xy \quad N(x,y) = x^2-y^2$

$\frac{\partial M}{\partial y} = -2x \quad \frac{\partial N}{\partial x} = 2x$

So, not exact. However, we can divide both sides by x^2 to get:

$$\left(1-\left(\frac{y}{x}\right)^2\right)y' = 2\left(\frac{y}{x}\right)$$

if we substitute $v = \frac{y}{x}$ $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$
we get:

$$(1-v^2)\left(v + x\frac{dv}{dx}\right) = 2v$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{2v}{1-v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{2v - v + v^3}{1-v^2} = \frac{v+v^3}{1-v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v+v^3}{1-v^2}$$

$$\Rightarrow \int\left(\frac{1-v^2}{v+v^3}\right)dv = \int\frac{dx}{x}$$

$$= \int\left(\frac{A}{v} + \frac{B}{1+v^2}\right)dv = \ln|x| + C$$

$$A(1+v^2) + B(v) = 1 - v^2$$

$$\Rightarrow A = 1 \quad B = -2v$$

$$\int\left(\frac{1}{v} - \frac{2v}{1+v^2}\right)dv = \ln|x| + C$$

$$\Rightarrow \ln|v| - \ln|1+v^2| = \ln|x| + C$$

$$\ln\left|\frac{v}{1+v^2}\right| = \ln|x| + C$$

$$\Rightarrow \quad \frac{v}{1+v^2} = Cx \Rightarrow v = Cx(1+v^2)$$

$$\Rightarrow \frac{y}{x} = Cx\left(1+\frac{y^2}{x^2}\right)$$

$$\Rightarrow \boxed{y = C(x^2+y^2)}$$

implicit is best we can do

19. $x^2y' + 2xy = 5y^3$

$\Rightarrow \quad y' + \frac{2}{x}y = \frac{5}{x^2}y^3$

this is a Bernoulli equation, and so we sabstitute:

$$v = y^{1-3} = \frac{1}{y^2}$$

$$\frac{dv}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y^3}{2}\frac{dv}{dx}$$

and so; we get:

$$-\frac{y^3}{2}\frac{dv}{dx} + \frac{2}{x}y = \frac{5y^3}{x^2}$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{4}{x}\frac{1}{y^2} = \frac{-10}{x^2}$$

assuming $x > 0$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{4}{x}v = \frac{-10}{x^2}$$

which is linear with $\rho(x) = e^{\int -\frac{4}{x}dx}$

$$\Rightarrow \quad \rho(x) = \frac{1}{x^4}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{v}{x^4}\right) = -\frac{10}{x^6}$$

$$\Rightarrow \quad \frac{v}{x^4} = \int \frac{-10}{x^6}\,dx = \frac{10}{5x^5} + C$$

So,

$$v = Cx^4 + \cancel{\tfrac{10}{3}x}\ \frac{2}{x}$$

and so

$$\frac{1}{y^2} = Cx^4 + \cancel{\tfrac{10}{3}x}\ \frac{2}{x}$$

$$\Rightarrow \quad y^2 = \frac{\cancel{3}x}{Cx^4 + \cancel{\tfrac{10}{3}x}\, 2} = \boxed{\cancel{\frac{3}{Cx^4+10x} = y^2}}$$

$$\boxed{y^2 = \frac{x}{cx^4+2}}$$

Note we assumed we're either dealing with $x > 0$ or $x < 0$, but not an interval with $x = 0$

31. Verify the given DE is exact, then solve it.

$$(2x+3y)\,dx + (3x+2y)\,dy = 0$$

$$M(x,y) = 2x+3y \qquad N(x,y) = 3x+2y$$

$$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$$

So, it's exact. Solving:

$$F(x,y) = \int (2x+3y)\,dx = x^2 + 3xy + C(y)$$

$$\frac{\partial F}{\partial y} = 3x + C'(y) = 3x + 2y \Rightarrow C'(y) = 2y \Rightarrow C(y) = y^2 + C$$

So

$$F(x,y) = x^2 + 3xy + y^2 + C$$

Or,

$$\boxed{x^2 + 3xy + y^2 = C}$$

43. Find a general solution to the reducible second order DE:

$$xy'' = y'$$

substitute $p = y'$, $p' = y''$

$\Rightarrow \; xp' = p$

$\Rightarrow \; x\frac{dp}{dx} = p \;\Rightarrow\; \int \frac{dp}{p} = \int \frac{dx}{x}$

$\Rightarrow \; \ln|p| = \ln|x| + C$

$\Rightarrow \; p = Cx$

$\Rightarrow \; \frac{dy}{dx} = Cx \;\Rightarrow\; \boxed{y(x) = Cx^2 + D}$

C, D constants

56. Suppose that $n \neq 0$ and $n \neq 1$. Show that the substitution $v = y^{1-n}$ transforms the Bernoulli equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

into the linear equation

$$\frac{dv}{dx} + (1-n)P(x)v(x) = (1-n)Q(x)$$

<u>Proof:</u>

$$v = y^{1-n}$$

$$\Rightarrow \; \frac{dv}{dx} = (1-n)y^{-n}\left(\frac{dy}{dx}\right)$$

and so,

$$\frac{dy}{dx} = \frac{y^n}{(1-n)} \frac{dv}{dx}$$

and if we plug this into our original equation we get:

$$\frac{y^n}{(1-n)} \frac{dv}{dx} + P(x)y = Q(x)y^n$$

$$\Rightarrow \quad \frac{dv}{dx} + (1-n)P(x)\,y^{1-n} = (1-n)\,Q(x)$$

$$\Rightarrow \quad \boxed{\frac{dv}{dx} + (1-n)\,P(x)\,v = (1-n)\,Q(x)}$$

Done!