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Math 2280

Assignment # 1 Solutions

1.1

Verify by substitution that the given function is a solution to the given ODE

1. $y' = 3x^2, \quad y = x^3+7$

$$y' = \frac{d}{dx}(x^3+7) = 3x^2 \checkmark$$

12. $x^2y'' - xy' + 2y = 0 \qquad y_1 = x\cos(\ln x)$
$\qquad y_2 = x\sin(\ln x)$

$$y_1' = -x\sin(\ln x)\frac{1}{x} + \cos(\ln x) = -\sin(\ln x) + \cos(\ln x)$$
$$y_1'' = -\cos(\ln x)\frac{1}{x} - \sin(\ln x)\frac{1}{x}$$

$$x^2y_1'' - xy' + 2y = -x\cos(\ln x) - x\sin(\ln x) + x\sin(\ln x) - x\cos(\ln x) + 2x\cos(\ln x)$$
$$= 0 \checkmark$$

$$y_2' = x\cos(\ln x)\frac{1}{x} + \sin(\ln x) = \cos(\ln x) + \sin(\ln x)$$
$$y_2'' = -\sin(\ln x)\frac{1}{x} + \cos(\ln x)\frac{1}{x}$$

$$x^2y_2'' - xy' + 2y = -x\sin(\ln x) + x\cos(\ln x) - x\cos(\ln x) - x\sin(\ln x) + 2x\sin(\ln x)$$
$$= 0 \checkmark$$

Substitute $y = e^{rx}$ into the given ODE and determine which values of r satisfy the ODE.

15.

$$y'' + y' - 2y = 0 \qquad \begin{aligned} y &= e^{rx} \\ y' &= re^{rx} \\ y'' &= r^2 e^{rx} \end{aligned}$$

$$\Rightarrow \quad r^2 e^{rx} + re^{rx} - 2e^{rx} = 0$$

$e^{rx} \neq 0$ for any r or x. So,

$(r^2 + r - 2)e^{rx} = 0$ implies $r^2 + r - 2 = 0$.

This happens for the two r values:

$$\frac{-1 \pm \sqrt{1 - 4(1)(-2)}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = \boxed{\{-2, 1\}}$$

Verify that $y(x)$ satisfies the given ODE. Then determine C so that it conforms to the given initial condition. Sketch $y(x)$ for many values of C, including the one satisfying the given initial data

20.

$$y' = x - y \qquad y(x) = Ce^{-x} + x - 1 \qquad y(0) = 10$$

$$y'(x) = -Ce^{-x} + 1 = x - (Ce^{-x} + x - 1) \checkmark$$

So, it works.

$y(0) = Ce^{-0} + 0 - 1 = 10$

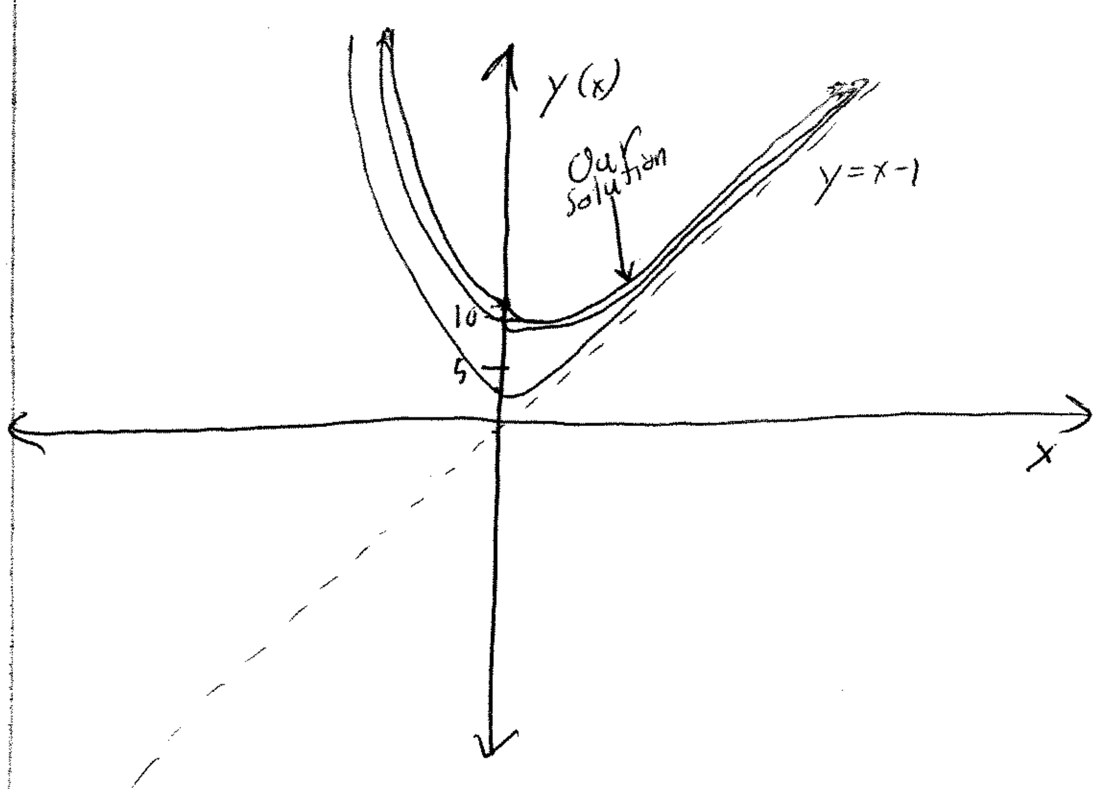
$\Rightarrow \quad C - 1 = 10$

$\Rightarrow \quad \boxed{C = 11}$

So, our particular solution is:

$$\boxed{y(x) = 11e^{-x} + x - 1}$$

Several Possible Solutions:



Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0)=2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P=10$ rodents. How long will it take for the population to grow to 100 rodents? To 1000? What's happening here?

45. $\frac{dP}{dt} = kP^2$

$\Rightarrow \int \frac{dP}{P^2} = \int k\,dt \Rightarrow -\frac{1}{P} = kt + C$

$\Rightarrow P(t) = \frac{-1}{kt + C}$

$P(0) = 2 = \frac{-1}{C}$ So, $C = -\frac{1}{2}$

$P(t) = \frac{1}{\frac{1}{2} - kt} = \frac{2}{1-2kt}$

When $P = 10$ we have:

$\cancel{10 = \frac{2}{1-kt} \Rightarrow 10 - 10kt = 2}$

$\cancel{\Rightarrow 8 = 10kt \Rightarrow}$

$\frac{dP}{dt} = 1 = k(10^2) \Rightarrow k = \frac{1}{100}$.

So,

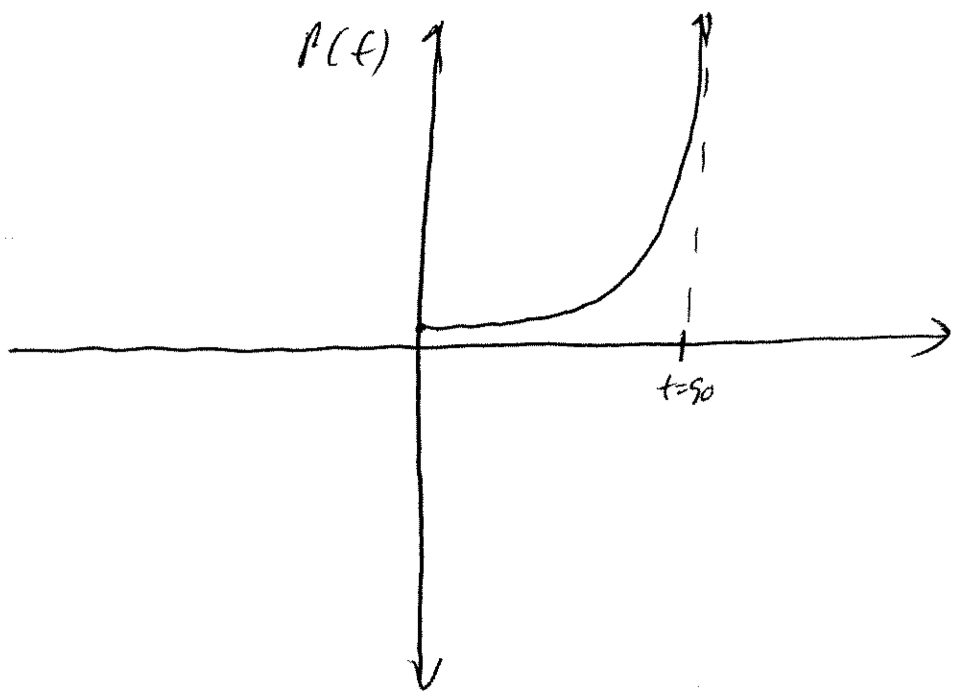
$$P(t) = \frac{2}{1 - {}^{2t}/_{100}} \Rightarrow P(t) = \frac{200}{100-2t} = \frac{100}{50-t}.$$

Time to 100:

$$100 = \frac{100}{50-t} \Rightarrow t = 49 \text{ months}$$

$$1{,}000 = \frac{100}{50-t} \Rightarrow t = 49.9 \text{ months.}$$

We have a vertial asymptote at $t = 50$ months.



1.2 Solve the differential equation:

1. $\frac{dy}{dx} = 2x+1 \; ; \quad y(0)=3$

$$\int dy = \int (2x+1)\,dx$$

$$\Rightarrow y(x) = x^2 + x + C$$

$$y(0) = 0^2 + 0 + C = 3 \quad \Rightarrow \quad C = 3$$

$$\boxed{y(x) = x^2 + x + 3}$$

6. $\frac{dy}{dx} = x\sqrt{x^2+9} \; ; \quad y(-4) = 0$

$$\int dy = \int x\sqrt{x^2+9}\,dx \qquad u = x^2+9 \qquad \frac{du}{2} = x\,dx$$

$$\cancel{y(x) = \tfrac{2}{3}\sqrt{}} \qquad \boxed{y(x) = \tfrac{1}{3}(x^2+9)^{3/2} + C}$$

Now, $y(-4) = \frac{1}{3}\left((-4)^2+9\right)^{3/2} + C$

$$= \frac{125}{3} + C = 0 \Rightarrow C = -125/3$$

So,

$$\boxed{y(x) = \frac{(x^2+9)^{3/2} - 125}{3}}$$

Solve for the position function $x(t)$.

11. $a(t) = 50 \quad v_0 = 10 \quad x_0 = 20$

$v(t) = 50t + C_1$

$v(0) = 50(0) + C_1 = v_0 = 10 \Rightarrow C_1 = 10$

$v(t) = 50t + 10$

$x(t) = 25t^2 + 10t + C_2$

$x(0) = 25(0)^2 + 10(0) + C_2 = C_2 = x_0 = 20$

$\Rightarrow \boxed{x(t) = 25t^2 + 10t + 20}$

15. $a(t) = 4(t+3)^2, \quad v_0 = -1, \quad x_0 = 1$

$\frac{dv}{dt} = 4(t+3)^2 \quad \int dv = \int 4(t+3)^2 dt$

$v(t) = \frac{4}{3}(t+3)^3 + C_1$

$v(0) = 36 + C_1 = -1 \Rightarrow C_1 = -37$

$\frac{dx}{dt} = \frac{4}{3}(t+3)^3 - 37$

$\Rightarrow x(t) = \frac{1}{3}(t+3)^4 - 37t + C_2$

$x(0) = 27 - 37(0) + C_2 = 1$

$\Rightarrow C_2 = -26$

$\boxed{x(t) = \frac{1}{3}(t+3)^4 - 37t - 26}$

27. A ball is thrown downward with an initial speed of 10 m/s from the top of a tall building. It strikes the ground at 60 m/s. How tall is the building?

$v(t) = at + 10$

so, the total time spent falling is:

$\Rightarrow v(t_f) = a t_f + 10 = 60$

$\Rightarrow t_f = \dfrac{50}{9.8}$ 9.8 = acceleration of gravity

The total distance traveled is:

$x(t) = \frac{1}{2} a t^2 + v_0 t$

$\Rightarrow x(t_f) = \frac{1}{2} a t_f^2 + 10 t_f = \boxed{178.6 m}$

35. A stone is dropped from rest at a height h above the Earth's surface. Show that the speed at which it hits the ground is $\sqrt{2gh}$.

The time, t_f, that it takes it to fall a distance h is:

$$h = x(t_f) = \frac{1}{2} g t_f^2 \Rightarrow t_f = \sqrt{2h/g}.$$

The velocity at time t_f is:

$$v(t_f) = g t_f = g\sqrt{2h/g} = \boxed{\sqrt{2gh}}$$

43.
Arthur Clarke's "The Wind from the Sun" (1963) describes Diana, a spacecraft propelled by the solar wind. It aluminum sail provides it with a constant acceleration of $0.001g = 0.0098\ m/s^2$. Suppose the spacecraft starts from rest at time $t=0$ and simultaneously fires a projectile (straight ahead in the same direction) that travels at one-tenth the speed of light $c = 3\times10^8\ m/s$. How long will it take the spaceship to catch up to the projectile, and how ~~long~~ far will it have traveled by then?

Distance spaceship travels:

$$x(t) = at^2 = (.001g)t^2 = (0.0098\ m/s^2)t^2$$

Distance projectile travels:

$$x(t) = vt = \frac{c}{10}t = (3\times10^7\ m/s)t$$

Equal when:

$$at^2 = vt \Rightarrow \cancel{a}\,t(at-v) = 0$$

or when $t = \frac{v}{a} = \boxed{3\times10^9 s}$

Distance traveled at that time:

$$x(3\times10^9 s) = (3\times10^7 m/s)(3\times10^9 s) = \boxed{9\times10^{16} m}$$

Using the projectile

1.3 Problems

In Problems 1 through 10, we have provided the slope field of the indicated differential equation, together with one or more solution curves. Sketch likely solution curves through the additional points marked in each slope field.

1. $\dfrac{dy}{dx} = -y - \sin x$

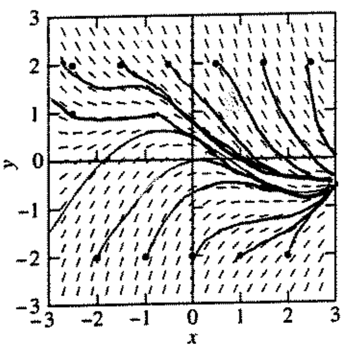


FIGURE 1.3.15.

2. $\dfrac{dy}{dx} = x + y$

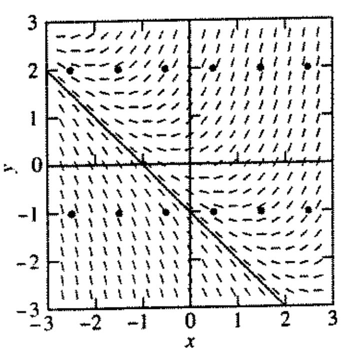


FIGURE 1.3.16.

3. $\dfrac{dy}{dx} = y - \sin x$

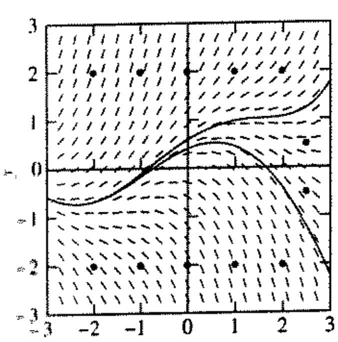


FIGURE 1.3.17.

4. $\dfrac{dy}{dx} = x - y$

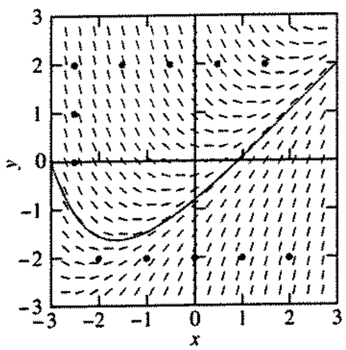


FIGURE 1.3.18.

5. $\dfrac{dy}{dx} = y - x + 1$

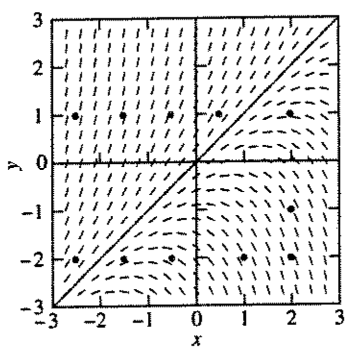


FIGURE 1.3.19.

6. $\dfrac{dy}{dx} = x - y + 1$

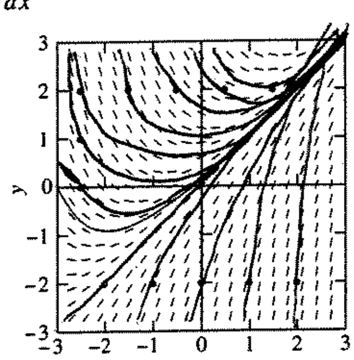


FIGURE 1.3.20.

7. $\dfrac{dy}{dx} = \sin x + \sin y$

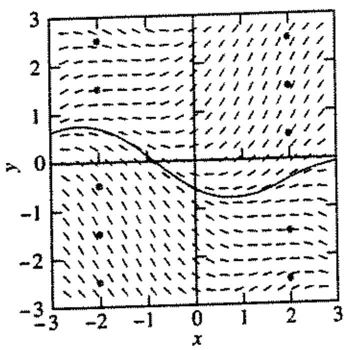


FIGURE 1.3.21.

8. $\dfrac{dy}{dx} = x^2 - y$

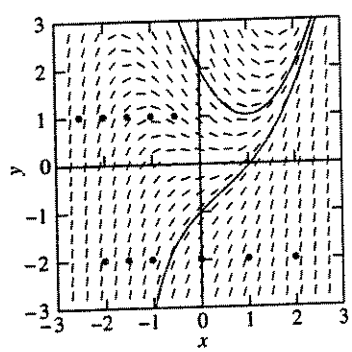


FIGURE 1.3.22.

9. $\dfrac{dy}{dx} = x^2 - y - 2$

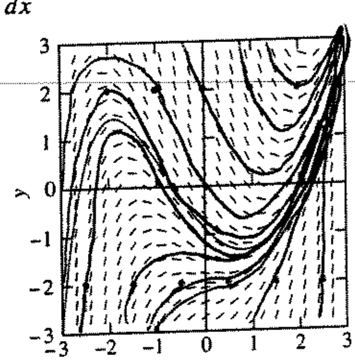


FIGURE 1.3.23.

10. $\dfrac{dy}{dx} = -x^2 + \sin y$

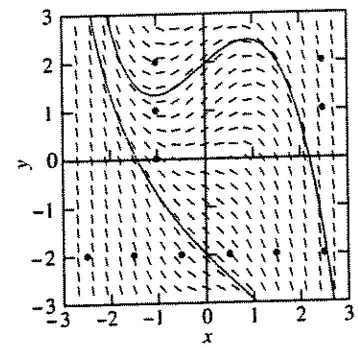


FIGURE 1.3.24.

In Problems 11 through 20, determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

11. $\dfrac{dy}{dx} = 2x^2y^2;\quad y(1) = -1$

12. $\dfrac{dy}{dx} = x \ln y;\quad y(1) = 1$

13. $\dfrac{dy}{dx} = \sqrt[3]{y};\quad y(0) = 1$

14. $\dfrac{dy}{dx} = \sqrt[3]{y};\quad y(0) = 0$

15. $\dfrac{dy}{dx} = \sqrt{x - y};\quad y(2) = 2$

16. $\dfrac{dy}{dx} = \sqrt{x - y};\quad y(2) = 1$

17. $y\dfrac{dy}{dx} = x - 1;\quad y(0) = 1$

18. $y\dfrac{dy}{dx} = x - 1;\quad y(1) = 0$

19. $\dfrac{dy}{dx} = \ln(1 + y^2);\quad y(0) = 0$

20. $\dfrac{dy}{dx} = x^2 - y^2;\quad y(0) = 1$

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

21. $y' = x + y,\quad y(0) = 0;\quad y(-4) = ?$

22. $y' = y - x,\quad y(4) = 0;\quad y(-4) = ?$

1.3 ~~#.~~

Determine whether theorem 1.3.1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether the theorem guarantees uniqueness.

11. $\frac{dy}{dx} = 2x^2y^2 = f(x,y)$ $\quad y(1)=-1$

$f(x,y)$, and $D_y f(x,y) = 4x^2y$ are continuous throughout $\mathbb{R}^2$. So, there exists a unique solution in a rectangle around $(1,-1)$.

15. $\frac{dy}{dx} = \sqrt{x-y} = f(y,y)$ $\quad y(2)=2$

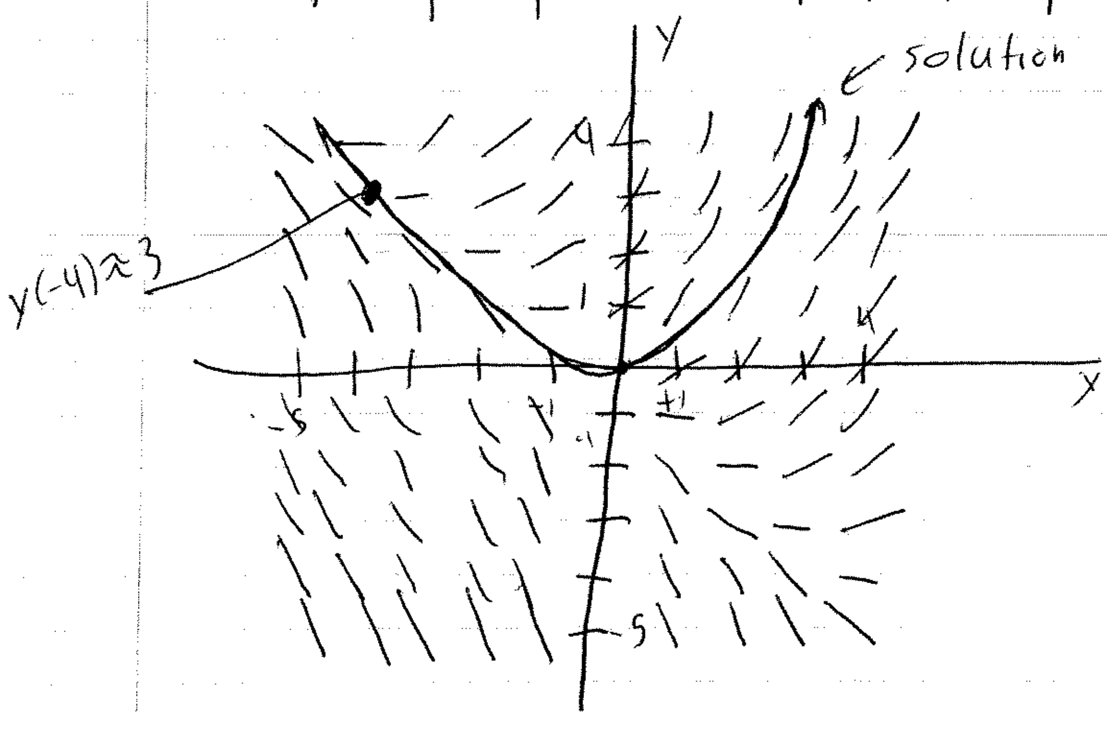
$$D_y f(x,y) = \frac{-1}{2\sqrt{x-y}}$$

Neither $f(x,y)$ nor $D_y f(x,y)$ are continuous in a rectangle that contains the point $(2,2)$. So, existance and uniqueness are not guaranteed.

21. Use the method of example 2 to construct a slope field for the given ODE. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

$y' = x + y, \quad y(0) = 0 \quad y(-4) = ?$

x+y	-5	-4	-3	-2	-1	0	1	2	3	4
-5	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
-4	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
-3	-8	-7	-6	-5	-4	-3	-2	-1	0	1
-2	-7	-6	-5	-4	-3	-2	-1	0	1	2
-1	-6	-5	-4	-3	-2	-1	0	1	2	3
0	-5	-4	-3	-2	-1	0	1	2	3	4
1	-4	-3	-2	-1	0	1	2	3	4	5
2	-3	-2	-1	0	1	2	3	4	5	6
3	-2	-1	0	1	2	3	4	5	6	7
4	-1	0	1	2	3	4	5	6	7	8



29. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leq c \\ (x-c)^3 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 3y^{2/3}$ for all x. Can you use the "left half" of the cubic $y = (x-c)^3$ in piecing together a solution curve of the differential equation? Is there a point (a,b) of the xy-plane such that the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ has either no solution or a unique solution that is defined for all x! Reconcile your answer with Theorem 1.

Verify:

$$y'(x) = 3(x-c)^2 = 3\left[(x-c)^3\right]^{2/3} = 3y^{2/3}$$

for $\cancel{x \leq c}$ $x > c$

$$y'(x) = 0 = 3[0]^{2/3} = 3y^{2/3}$$

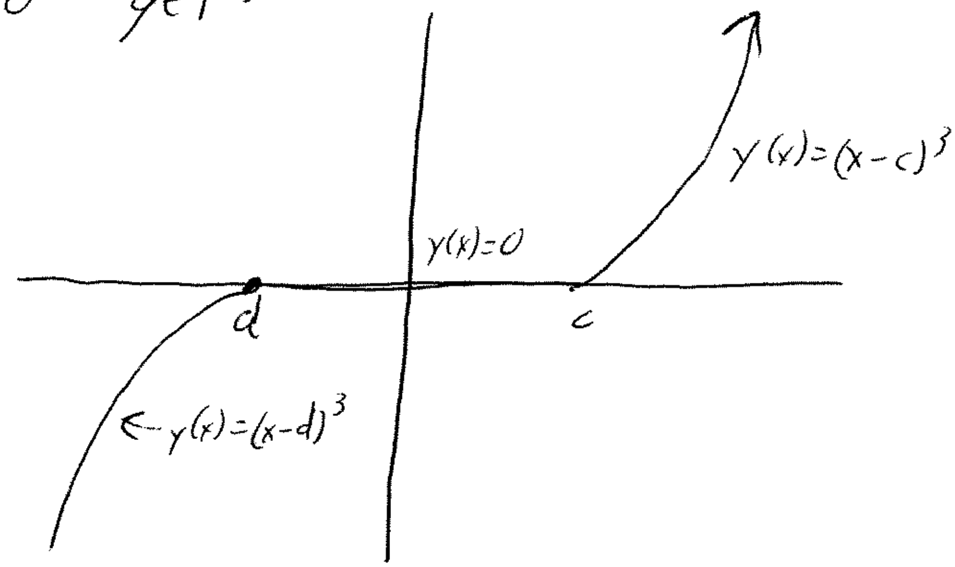
for $\cancel{x > c}$ $x \leq c$

at $x = c$ $y'(x) = 0$ so it patches up

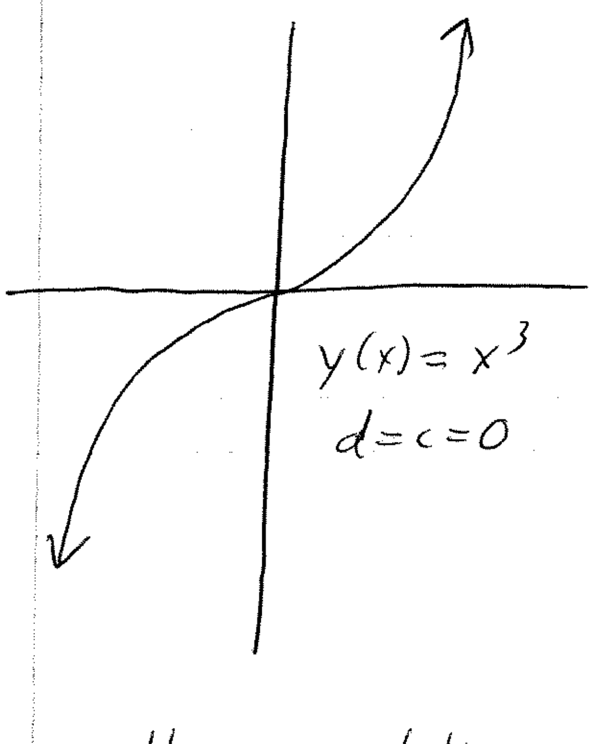
We can construct any other solution curve as:

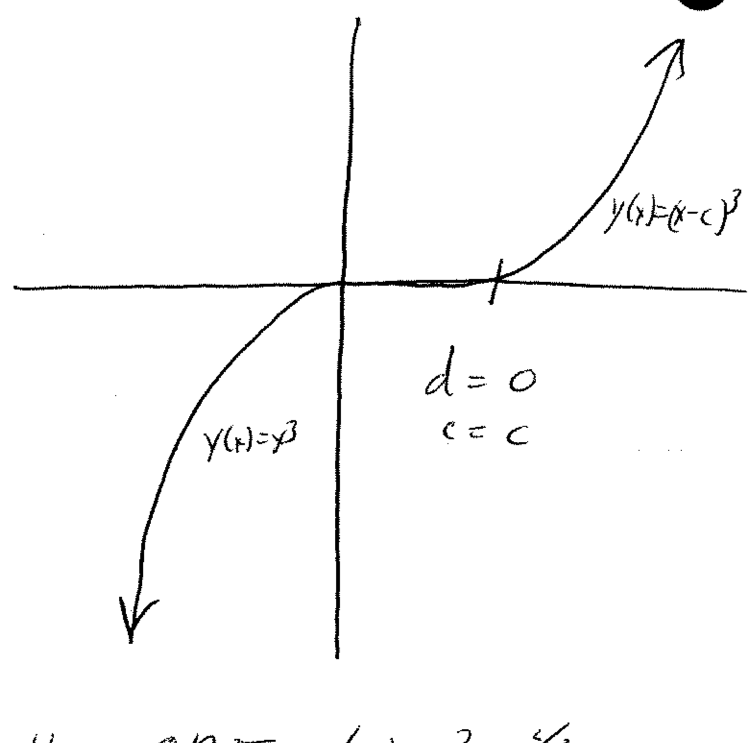
$$y(x) = \begin{cases} (x-d)^3 & \text{for } x < d \\ 0 & \text{for } d \leq x \leq c \\ (x-c)^3 & \text{for } x > c \end{cases}$$

to get:



any solution of this form satisfies the given ODE. Other curves:





all are solutions to the ODE $y'(x)=3y^{2/3}$

There is no point where ~~f(~~ $y' = 3y^{2/3}$ has either no solution or a solution defined for all x. If $b < 0$ then that fixes our choice of d, but not c. If $b > 0$ this fixes our choice of c, but not d. If $b = 0$ then we can choose any c and d provided $d \leq a \leq c$.

This works within theorem 1, as if $b \neq 0$ our solution will be unique within a rectangle around (a, b). If $b = 0$ then:

$$D_y f(x,y) = \frac{2}{y^{1/3}}$$

is not continuous at $y = 0$. So, we're not guaranteed a unique solution, and we don't have one