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Math 2280
Solutions

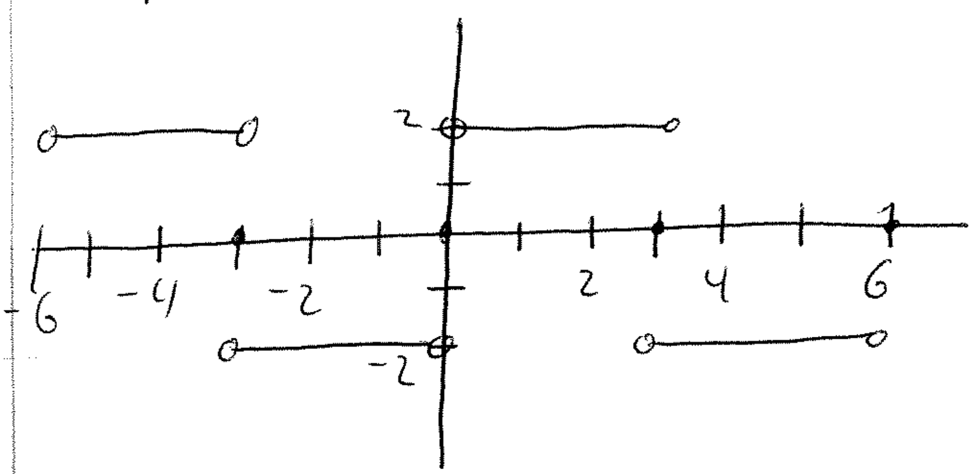
Assignment #14

9.2.1. Sketch the graph of $f(t)$ and find its Fourier series:

$$f(t) = \begin{cases} -2 & -3 < t < 0 \\ 2 & 0 < t < 3 \end{cases}$$

and is 6 periodic with the points of discontinuity satisfying the "average" condition

<u>Graph:</u>



Now, $f(t)$ is odd, so all the cosine terms will be 0. The sine terms are:

$n \geq 1$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(t)\sin\left(\frac{n\pi t}{3}\right)dt$$

Now, as $f(t)$ is odd, $f(t)\sin\left(\frac{n\pi t}{3}\right)$ is even, and so

$$b_n = \frac{2}{3}\int_0^3 2\sin\left(\frac{n\pi t}{3}\right)$$

$$= \frac{4}{3}\left(\frac{3}{n\pi}\right)\left(-\cos\left(\frac{n\pi t}{3}\right)\right)\Big|_0^3$$

$$= \frac{4}{n\pi}\left(-\cos(n\pi) - (-1)\right)$$

$$= \frac{4}{n\pi}\left(1 - \cos(n\pi)\right)$$

Now, $\cos(n\pi) = (-1)^n$.

So, $$b_n = \begin{cases} \frac{8}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

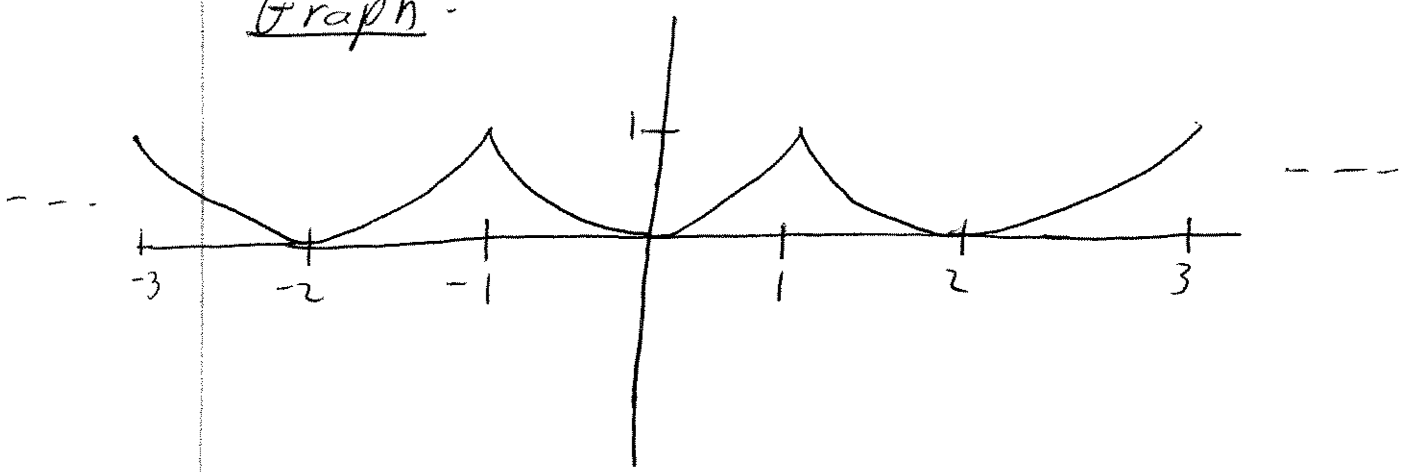
and so,

$$FT(f(t)) = \frac{8}{\pi}\left(\sin\left(\frac{\pi t}{3}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{3}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{3}\right) + \cdots\right)$$

1.2-9

$f(t) = t^2 \quad , -1 < t < 1$

<u>Graph</u>:



We note that $f(t)$ is even, so all the sine terms will be 0 and we'll only have cosine terms.

$$a_0 = \int_{-1}^{1} t^2 \, dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

for $n \geq 1$

$$a_n = \int_{-1}^{1} t^2 \cos(n\pi t)\, dt$$

$$= 2\int_{0}^{1} t^2 \cos(n\pi t)\, dt$$

as $t^2 \cos(n\pi t)$ is even.

$$2\int_0^1 t^2\cos(n\pi t)\,dt \qquad \begin{array}{l} u = t^2 \\ du = 2t\,dt \end{array} \qquad \begin{array}{l} dv = \cos(n\pi t)\,dt \\ v = \frac{\sin(n\pi t)}{n\pi} \end{array}$$

$$= 2\,\frac{t^2\sin(n\pi t)}{n\pi}\Big|_0^1 - \frac{4}{n\pi}\int_0^1 t\sin(n\pi t)\,dt$$

$$= -\frac{4}{n\pi}\int_0^1 t\sin(n\pi t)\,dt \qquad \begin{array}{l} u = t \\ du = dt \end{array} \qquad \begin{array}{l} dv = \sin(n\pi t) \\ v = -\frac{\cos(n\pi t)}{n\pi} \end{array}$$

$$= \frac{4}{n^2\pi^2}\,t\cos(n\pi t)\Big|_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \cos(n\pi t)\,dt$$

$$= \frac{4}{n^2\pi^2}\cos(n\pi) + \frac{4}{n^3\pi^3}\sin(n\pi t)\Big|_0^1$$

$$= \frac{4}{n^2\pi^2}(-1)^n.$$

So, $\qquad a_n = \frac{(-1)^n 4}{n^2\pi^2} \qquad n \geq 1$

and

$$\boxed{FT(f(t)) = \frac{1}{3} - \frac{4}{\pi^2}\left(\cos(\pi t) - \frac{\cos(2\pi t)}{2^2} + \frac{\cos(3\pi t)}{3^2} - \cdots\right)}$$

2.15 a) Suppose that f is a function of period 2π with $f(t) = t^2$ for $0 < t < 2\pi$. Show that

$$f(t) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}$$

and sketch the graph of f, indicating the value at each discontinuity.

For our Fourier series we get

$$a_0 = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(t)\,dt = \frac{1}{2\pi}\left[\int_{-2\pi}^{0} (t+2\pi)^2\,dt + \int_0^{2\pi} t^2\,dt\right]$$

From Symmetry $= \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{t^3}{3\pi}\Big|_0^{2\pi} = \frac{8\pi^2}{3}$

for $n \geq 1$

$$a_n = \frac{1}{2\pi}\left[\int_{-2\pi}^{0} (t+2\pi)^2 \cos\left(\frac{nt}{2}\right)dt + \int_0^{2\pi} t^2 \cos\left(\frac{nt}{2}\right)dt\right]$$

$u = t + 2\pi$
$du = dt$
$\Rightarrow \int_{-2\pi}^{0} (t+2\pi)^2 \cos\left(\frac{nt}{2}\right)dt$

$$= \int_0^{2\pi} u^2 \cos\left(\frac{n(u-2\pi)}{2}\right)du$$

$$= \int_0^{2\pi} u^2 \cos\left(\frac{nu}{2}\right)(-1)^n\,du$$

So,

$$a_n = \frac{1}{2\pi}\left[\int_0^{2\pi} (-1)^n u^2 \cos\left(\frac{nu}{2}\right) du + \int_0^{2\pi} t^2 \cos\left(\frac{nt}{2}\right) dt\right]$$

$= 0$ when n is odd and

$$= \frac{1}{\pi}\int_0^{2\pi} t^2 \cos\left(\frac{nt}{2}\right) dt$$

$u = t^2 \qquad dv = \cos\left(\frac{nt}{2}\right) dt$

$du = 2t\,dt \qquad v = \frac{2}{n}\sin\left(\frac{nt}{2}\right)$

$$= \frac{2t^2}{\pi n}\sin\left(\frac{nt}{2}\right)\Big|_0^{2\pi} - \frac{4}{n\pi}\int_0^{2\pi} t\sin\left(\frac{nt}{2}\right) dt$$

$$= 0 - \frac{4}{n\pi}\int_0^{2\pi} t\sin\left(\frac{nt}{2}\right) dt$$

$u = t \qquad dv = \sin\left(\frac{nt}{2}\right) dt$

$du = dt \qquad v = -\frac{2}{n}\cos\left(\frac{nt}{2}\right)$

$$= \frac{8}{n^2\pi} t\cos\left(\frac{nt}{2}\right)\Big|_0^{2\pi} - \frac{8}{n^2\pi}\int_0^{2\pi} \cos\left(\frac{nt}{2}\right) dt$$

$$= \frac{8}{n^2\pi} 2\pi\cos(n\pi) - \frac{16}{n^3\pi}\sin\left(\frac{nt}{2}\right)\Big|_0^{2\pi}$$

$= \frac{16}{n^2}$ when n is even.

For the sine terms we have:

$$b_n = \frac{1}{2\pi}\left[\int_{-2\pi}^{0}(t+2\pi)^2 \sin\left(\frac{n\cancel{\pi}t}{2}\right)dt + \int_{0}^{2\pi} t^2 \sin\left(\frac{nt}{2}\right)dt\right]$$

$u = t+2\pi$
$du = dt$

$$\int_{-2\pi}^{0}(t+2\pi)^2 \sin\left(\frac{nt}{2}\right)dt = \int_{0}^{2\pi} u^2 \sin\left(\frac{n(u-2\pi)}{2}\right)$$

$$= \int_{0}^{2\pi}(-1)^n u^2 \sin\left(\frac{nu}{2}\right)du$$

So,

$$b_n = \frac{1}{2\pi}\left[\int_{0}^{2\pi}(-1)^n u^2 \sin\left(\frac{nu}{2}\right)du + \int_{0}^{2\pi} t^2 \sin\left(\frac{nt}{2}\right)dt\right]$$

$= 0$ when n is odd.

$$= \frac{1}{\pi}\int_{0}^{2\pi} t^2 \sin\left(\frac{nt}{2}\right)dt \quad \text{when } n \text{ is even}$$

$u = t^2 \qquad dv = \sin\left(\frac{nt}{2}\right)dt$
$du = 2t\,dt \qquad v = -\frac{2}{n}\cos\left(\frac{nt}{2}\right)$

$$= \frac{-2t^2}{\pi n}\cos\left(\frac{nt}{2}\right)\Big|_{0}^{2\pi} + \frac{4}{\pi n}\int_{0}^{2\pi} t\cos\left(\frac{nt}{2}\right)dt$$

$$= \frac{-2t^2}{\pi n}\left((-1)^n - 1\right) + \frac{4}{\pi n}\int_{0}^{2\pi} t\cos\left(\frac{nt}{2}\right)dt$$

$$= \frac{4}{\pi n}\int_{0}^{2\pi} t\cos\left(\frac{nt}{2}\right)dt \quad \text{when } n \text{ is even.}$$

$$b_n = \frac{4}{n\pi}\int_0^{2\pi} t\cos\left(\frac{nt}{2}\right)dt$$

$$u = t \qquad dv = \cos\left(\frac{nt}{2}\right)dt$$
$$du = dt \qquad v = \frac{2}{n}\sin\left(\frac{nt}{2}\right)$$

$$= \frac{8t}{n^2\pi}\sin\left(\frac{nt}{2}\right)\Big|_0^{2\pi} - \frac{8}{n^2\pi}\int_0^{2\pi}\sin\left(\frac{nt}{2}\right)dt$$

$$= 0 + \frac{16}{n^3\pi}\cos\left(\frac{nt}{2}\right)\Big|_0^{2\pi}$$

$$= \frac{16}{n^3\pi}\left((-1)^n - 1\right) = 0 \text{ when } n \text{ is even.}$$

So, $b_n = 0$ for all n.

therefore,

$$FT(f(t)) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{16\cos(2nt)}{(2n)^2}$$

$$= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos(2nt)}{n^2}$$

Now, $\cos(2nt) =$

1-2.15 a) Suppose that f is a function of period 2π with $f(t)=t^2$ for $0<t<2\pi$. Show that

$$f(t)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^2}-4\pi\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}$$

and sketch the graph of f, indicating the value at each discontinuity.

Taking the Fourier transform we have:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(t+2\pi)^2\,dt+\frac{1}{\pi}\int_{0}^{\pi}t^2\,dt$$

$u=t+2\pi$
$du=dt$

$$=\frac{1}{\pi}\int_{\pi}^{2\pi}u^2\,du+\frac{1}{\pi}\int_{0}^{\pi}t^2\,dt$$

$$=\frac{1}{\pi}\int_{0}^{2\pi}t^2\,dt=\frac{t^3}{3\pi}\Big|_0^{2\pi}=\frac{8\pi^2}{3}$$

For $n\geq 1$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos\left(\frac{nt}{\cancel{2}}\right)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(t+2\pi)^2\cos\left(\frac{nt}{\cancel{2}}\right)dt + \frac{1}{\pi}\int_{0}^{\pi}t^2\cos\left(\frac{nt}{\cancel{2}}\right)$$

$u = t+2\pi$
$du = dt$

$$= \frac{1}{\pi}\int_{\pi}^{2\pi}u^2\cos\left(\frac{n(u-2\pi)}{\cancel{2}}\right)d\cancel{t}u + \frac{1}{\pi}\int_{0}^{\pi}t^2\cos\left(\frac{nt}{\cancel{2}}\right)dt$$

$$= \frac{1}{\pi}\int_{\pi}^{2\pi}u^2\cos(nu)\,du + \frac{1}{\pi}\int_{0}^{\pi}t^2\cos(nt)\,dt$$

$$= \frac{1}{\pi}\int_{0}^{2\pi}t^2\cos(nt)\,dt$$

$u = t^2 \qquad dv = \cos(nt)\,dt$
$du = 2t\,dt \qquad v = \frac{\sin(nt)}{n}$

$$= \left.\frac{t^2\sin(nt)}{\pi n}\right|_0^{2\pi} - \frac{2}{\pi n}\int_0^{2\pi}t\sin(nt)\,dt$$

$$= 0 - \frac{2}{\pi n}\int_0^{2\pi}t\sin(nt)\,dt$$

$u = t \qquad dv = \sin(nt)\,dt$
$du = dt \qquad v = -\cos(nt)/n$

$$= \left.\frac{2t\cos(nt)}{\pi n^2}\right|_0^{2\pi} - \frac{2}{\pi n^2}\int_0^{2\pi}\cos(nt)\,dt$$

$$= \frac{4}{n^2} - \left.\frac{2}{\pi n^3}\sin(nt)\right|_0^{2\pi} = \frac{4}{n^2}$$

As for the sine terms we get:

$$b_n = \frac{1}{\pi} \int_{\cancel{0}-\pi}^{\cancel{2}\pi} f(t) \sin(nt)\, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (t+2\pi)^2 \sin(nt)\, dt + \frac{1}{\pi} \int_{0}^{\pi} t^2 \sin(nt)\, dt$$

using identical reasoning as last time:

$$= \frac{1}{\pi} \int_{0}^{2\pi} t^2 \sin(nt)\, dt$$

$u = t^2 \quad dv = \sin(nt)\,dt$
$du = 2t\,dt \quad v = -\cos(nt)/n$

$$= -\frac{t^2 \cos(nt)}{n\pi} \Big|_0^{2\pi} + \frac{2}{n\pi} \int_0^{2\pi} t \cos(nt)\, dt$$

$$= \frac{-4\pi}{n} + \frac{2}{n\pi} \int_0^{2\pi} t \cos(nt)\, dt$$

$u = t \quad dv = \cos(nt)$
$du = dt \quad v = \sin(nt)/n$

$$= -\frac{4\pi}{n} + \frac{2t\sin(nt)}{n^2\pi} \Big|_0^{2\pi} - \frac{2}{n^2\pi} \int_0^{2\pi} \sin(nt)\, dt$$

$$= -\frac{4\pi}{n} + 0 - 0 + \frac{2}{n^3\pi} \cos(nt) \Big|_0^{2\pi}$$

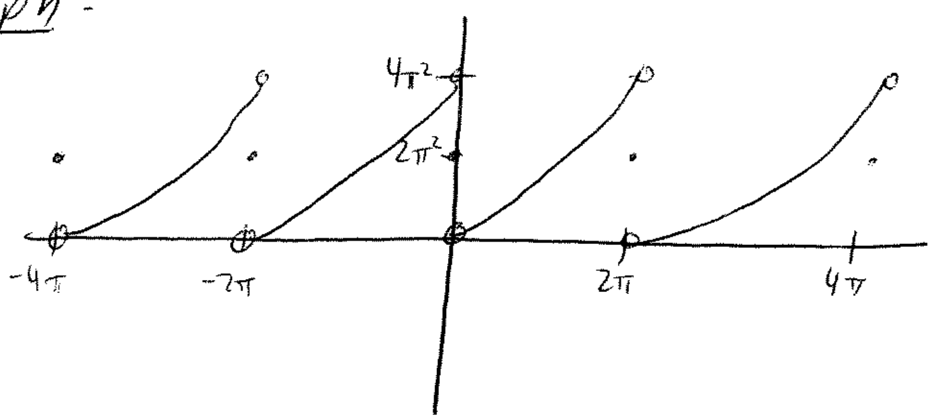
$$= -\frac{4\pi}{n} + \frac{2}{n^3\pi}(1-1) = -\frac{4\pi}{n}.$$

So, the Fourier transform is indeed:

$$FT(f(t)) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}$$

where $\frac{4\pi^2}{3} = \frac{a_0}{2}$.

<u>Graph</u>:



b) Deduce the series summations in Eqs. (16) and (17) from the above Fourier series

Plug in $t=0$ we get:

$$2\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow 4\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Plug in $t=\pi$ to get:

$$\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$$

1-2-17 a) Suppose that f is a function of period 2 with $f(t) = t$ for $0 < t < 2$. Show that

$$f(t) = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi t)}{n}$$

and sketch the ~~val~~ graph of f, indicating the value at each discontinuity.

$$a_0 = \int_0^2 f(t)\,dt = \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = \frac{4}{2} - 0 = 2.$$

for $n \geq 1$

$$a_n = \int_0^2 f(t)\cos(n\pi t)\,dt = \int_0^2 t\cos(n\pi t)\,dt$$

$u = t \quad dv = \cos(n\pi t)$

$du = dt \quad v = \frac{\sin(n\pi t)}{n\pi}$

$$\Rightarrow a_n = \frac{t\sin(n\pi t)}{n\pi}\Big|_0^2 - \frac{1}{n\pi}\int_0^2 \sin(n\pi t)\,dt$$

$$= 0 + \frac{\cos(n\pi t)}{n^2\pi^2}\Big|_0^2 = \frac{1}{n^2\pi^2}(1-1) = 0.$$

$$b_n = \int_0^2 f(t)\sin(n\pi t)\,dt = \int_0^2 t\sin(n\pi t)\,dt$$

$u = t \qquad dv = \sin(n\pi t)$

$du = dt \qquad v = -\cos(n\pi t)/n\pi$

$$\Rightarrow b_n = -\frac{t\cos(n\pi t)}{n\pi}\Big|_0^2 + \frac{1}{n\pi}\int_0^2 \cos(n\pi t)\,dt$$

$$= \frac{-2}{n\pi} + \frac{1}{n^2\pi^2}\sin(n\pi t)\Big|_0^2$$

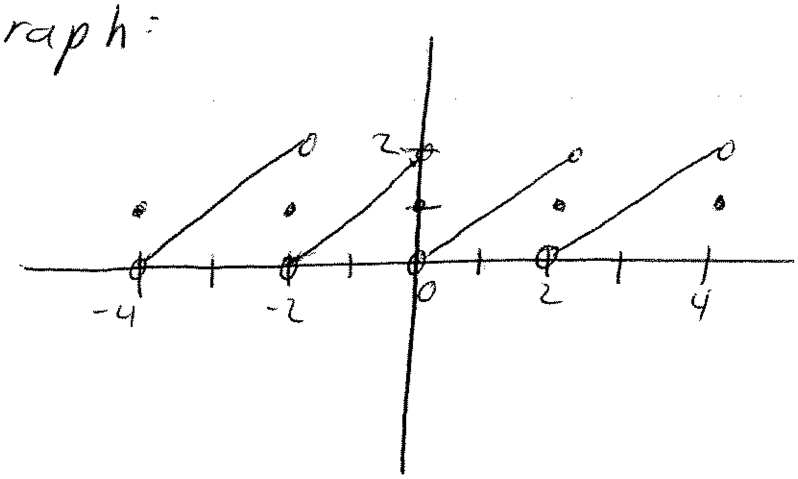
$$= \frac{-2}{n\pi}.$$

So, the Fourier ~~tra~~ series is:

$$FT(f(t)) = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi t)}{n}$$

and $1 = a_0/2 = 2/2$.

Graph:



b) Substitute an appropriate value of t to deduce Leibniz's series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

If we plug in $t = \frac{1}{2}$ into the series from part a) we get:

$$\frac{1}{2} = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n}$$

$$\Rightarrow \quad \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n}$$

$$\Rightarrow \quad \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$$

9.2.20 Derive the Fourier series:

$$\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2} = \frac{3t^2 - 6\pi t + 2\pi^2}{12}$$

$0 < t < 2\pi$

We note:

$$\int_0^{2\pi} \cos(nt)dt = \frac{\sin(nt)}{n}\Big|_0^{2\pi} = 0$$

if $n \neq 0$.

If $n = 0$ $\int_0^{2\pi} dt = 2\pi$

$$\int_0^{2\pi} \sin(nt)dt = \frac{-\cos(nt)}{n}\Big|_0^{2\pi} = \frac{-1}{n} - \left(\frac{-1}{n}\right) = 0.$$

$$\int_0^{2\pi} t\cos(nt)dt = \frac{t\sin(nt)}{n} - \frac{\cos(nt)}{n^2}\Big|_0^{2\pi}$$

$$= \left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{1}{n^2}\right) = 0.$$

if $n \neq 0$.

If $n = 0$

$$\int_0^{2\pi} t\,dt = \frac{t^2}{2} = 2\pi^2$$

$$\int_0^{2\pi} t \sin(nt)\,dt = \frac{-t\cos(nt)}{n} + \frac{\sin(nt)}{n^2}\Bigg|_0^{2\pi}$$

$$= \left(\frac{-2\pi}{n} + 0\right) - \left(0 + 0\right) = \frac{-2\pi}{n}$$

and

$$\int_0^{2\pi} t^2 \cos(nt)\,dt = \frac{t^2 \sin(nt)}{n} + \frac{2t\cos(nt)}{n^2} - \frac{2\sin(nt)}{n^3}\Bigg|_0^{2\pi}$$

$$= \frac{4\pi}{n^2}$$

if $n \neq 0$.

If $n = 0$

$$\int_0^{2\pi} t^2\,dt = \frac{t^3}{3}\Bigg|_0^{2\pi} = \frac{8\pi^3}{3}$$

and

$$\int_0^{2\pi} t^2 \sin(nt)\,dt = -\frac{t^2\cos(nt)}{n} + \frac{2t\sin(nt)}{n^2} + \frac{2\cos(nt)}{n^3}\Bigg|_0^{2\pi}$$

$$= \left(\frac{-4\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{2}{n^3}\right) = -\frac{4\pi^2}{n}$$

So, to recap:

For $n=0$ we have:

$$\int_0^{2\pi} dt = 2\pi \qquad \int_0^{2\pi} t\,dt = 2\pi^2 \qquad \int_0^{2\pi} t^2 dt = \frac{8\pi^3}{3}$$

For $n \neq 0$ we have:

$$\int_0^{2\pi} \cos(nt)\,dt = 0 \qquad \int_0^{2\pi} t\cos(nt)\,dt = 0 \qquad \int_0^{2\pi} t^2\cos(nt)\,dt = \frac{4\pi}{n^2}$$

and so:

$$\int_0^{2\pi} \frac{3t^2 - 6\pi t + 2\pi^2}{12} = \frac{3}{12}\left(\frac{8\pi^3}{3}\right) - \frac{6\pi}{12}\left(2\pi^2\right) + \frac{2\pi^2}{12}\left(2\pi\right)$$

$$= \frac{2\pi^3}{3} - \pi^3 + \frac{\pi^3}{3} = 0.$$

for $n \neq 0$ we have:

$$\int_0^{2\pi}\left(\frac{3t^2 - 6\pi t + 2\pi^2}{12}\right)\cos(nt)\,dt = \frac{3}{12}\left(\frac{4\pi}{n^2}\right) = \frac{\pi}{n^2}$$

So, the Fourier coefficients are:

$$a_0 = 0 \qquad a_n = \frac{1}{\pi}\left(\frac{\pi}{n^2}\right) = \frac{1}{n^2}$$

Now, the sine coefficients are:

$$\int_0^{2\pi} \sin(nt)dt = 0 \quad \int_0^{2\pi} t\sin(nt)dt = -\frac{2\pi}{n} \quad \int_0^{2\pi} t^2\sin(nt)dt = \frac{-4\pi^2}{n}$$

So,

$$\frac{1}{\pi} \int_0^{2\pi} \left(\frac{3t^2 - 6\pi t + 2\pi^2}{12}\right) \sin(nt)dt$$

$$= \frac{3}{12\pi}\left(-\frac{4\pi^2}{n}\right) - \frac{6}{12\cancel{\pi}}\left(-\frac{2\pi}{n}\right) = -\frac{\pi}{n} + \frac{\pi}{n} = 0$$

So, $b_n = 0$.

Therefore,

$$FT\left(\frac{3t^2 - 6\pi t + 2\pi^2}{12}\right) = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$$

as $a_n = \frac{1}{n^2}$ for $n \neq 0$ $a_0 = 0$ and $b_n = 0$.

9.3.1. Find the cosine and sine series of f and sketch the graphs of the two extensions of f to which these two series converge.

$$f(t) = 1 \qquad 0 < t < \pi$$

The sine series:

$$b_n = \frac{2}{\cancel{2}\pi} \int_0^{\pi} \sin(nt)\,dt = \frac{-2\cos(nt)}{n\pi}\Big|_0^{\pi}$$

$$= \frac{-2}{n\pi}\left((-1)^{n} - 1\right)$$

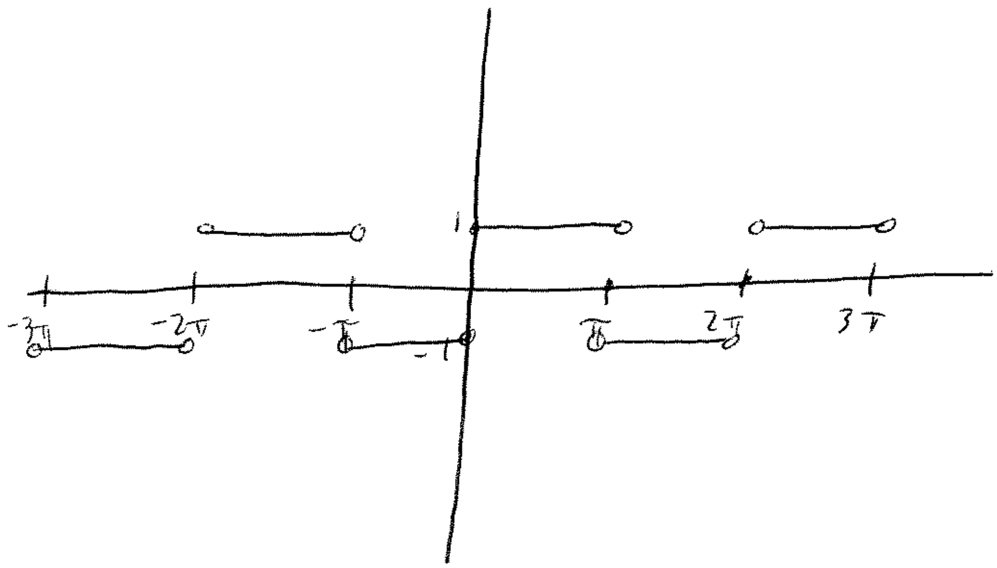
$$= \begin{cases} \dfrac{4}{n\pi} & \text{if } n \text{ is } \cancel{\text{even}} \text{ odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

So,

$$f(t) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left((2n+1)t\right)$$

is the sine series.

The function it represents is:



The cosine series is:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} dt = 2$$

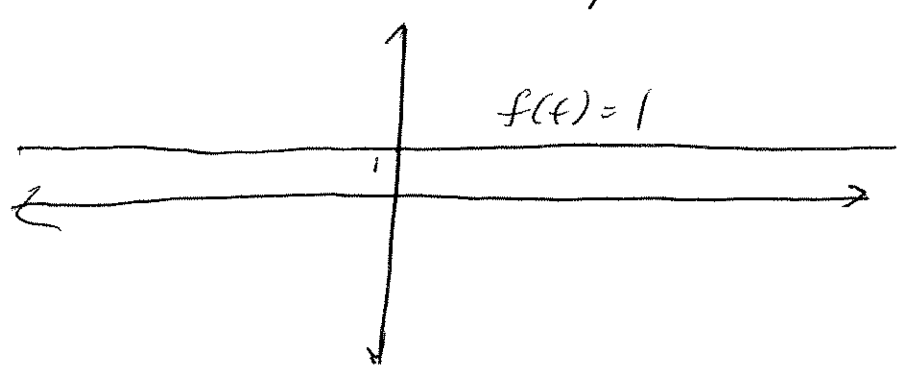
$n \geq 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos(nt)\,dt = \frac{2}{\pi}\left(\frac{\sin(nt)}{n}\right)\Big|_0^{\pi} = 0.$$

So,

$$f(t) = \frac{2}{2} + 0 = 1.$$

The function it represents is:



9.3.5.

$$f(t) = \begin{cases} 0, & 0 < t < 1; \\ 1, & 1 < t < 2; \\ 0. & 2 < t < 3. \end{cases}$$

cosine series

$$a_0 = \frac{2}{3}\int_0^3 f(t)\,dt = \frac{2}{3}\int_1^2 dt = \frac{2}{3}$$

$n \geq 1$

$$a_n = \frac{2}{3}\int_0^3 f(t)\cos\left(\frac{n\pi t}{3}\right)dt$$

$$= \frac{2}{3}\int_1^2 \cos\left(\frac{n\pi t}{3}\right)dt$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi t}{3}\right)\Bigg|_1^2$$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right]$$

$$\sin\left(\frac{2n\pi}{3}\right) = \begin{cases} +\frac{\sqrt{3}}{2} & \text{for } n = 3k+1 \\ -\frac{\sqrt{3}}{2} & \text{for } n = 3k+2 \\ 0 & \text{for } n = 3k \end{cases}$$

$$\sin\left(\frac{n\pi}{3}\right) = \begin{cases} \cancel{1}\,\frac{\sqrt{3}}{2} & \text{for } n = 6k+1 \\ \frac{\sqrt{3}}{2} & \text{for } n = 6k+2 \\ 0 & \text{for } n = 6k+3 \\ -\frac{\sqrt{3}}{2} & \text{for } n = 6k+4 \\ -\frac{\sqrt{3}}{2} & \text{for } n = 6k+5 \end{cases}$$

So,

$$\frac{2}{n\pi}\left[\sin\left(\frac{2n\pi}{3}\right)-\sin\left(\frac{n\pi}{3}\right)\right]$$

$$=\begin{cases} 0 & \text{for } n = 6k+1 \\ -\frac{2\sqrt{3}}{n\pi} & \text{for } n = 6k+2 \\ 0 & \text{for } n = 6k+3 \\ \frac{2\sqrt{3}}{n\pi} & \text{for } n = 6k+4 \\ 0 & \text{for } n = 6k+5 \\ 0 & \text{for } n = 6k \end{cases}$$

So, the cosine series is:

$$f(x) = \frac{1}{3} - \frac{2\sqrt{3}}{\pi}\left(\frac{1}{2}\cos\left(\frac{2\pi t}{3}\right) - \frac{1}{4}\cos\left(\frac{4\pi t}{3}\right) + \frac{1}{8}\cos\left(\frac{8\pi t}{3}\right) - \frac{1}{10}\cos\left(\frac{10}{3}\right. \right.$$

$$\left. \cdots \right)$$

The sine series is:

$$b_n = \frac{2}{3}\int_0^3 f(t)\sin\left(\frac{n\pi t}{3}\right)dt$$

$$= \frac{2}{3}\int_1^2 \sin\left(\frac{n\pi t}{3}\right)dt$$

$$= \frac{-2}{n\pi}\cos\left(\frac{n\pi t}{3}\right)\Big|_1^2$$

$$= -\frac{2}{n\pi}\left(\cos\left(\frac{2n\pi}{3}\right)-\cos\left(\frac{n\pi}{3}\right)\right)$$

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Now,

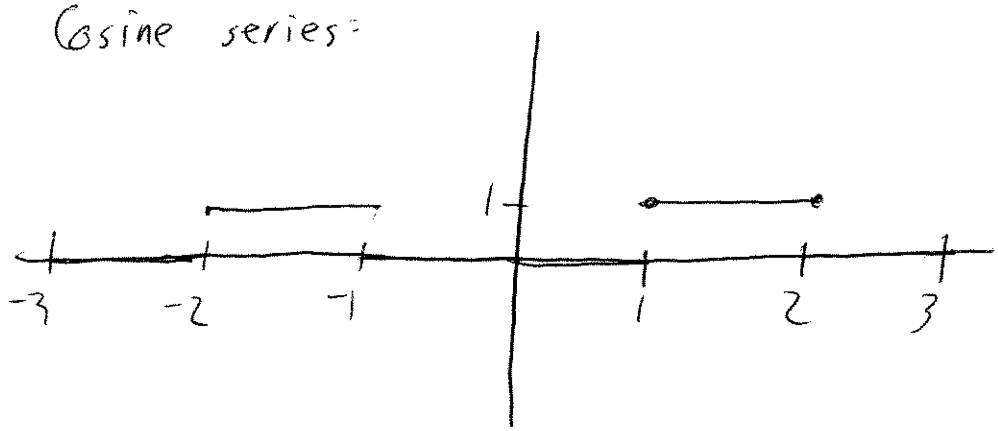
$$\cos\left(\frac{2n\pi}{3}\right) - \cos\left(\frac{n\pi}{3}\right) = \begin{cases} -1 & \text{for } n = 6k+1 \\ 0 & \text{for } n = 6k+2 \\ 2 & \text{for } n = 6k+3 \\ 0 & \text{for } n = 6k+4 \\ -1 & \text{for } n = 6k+5 \\ 0 & \text{for } n = 6k \end{cases}$$

So, the sine series is:

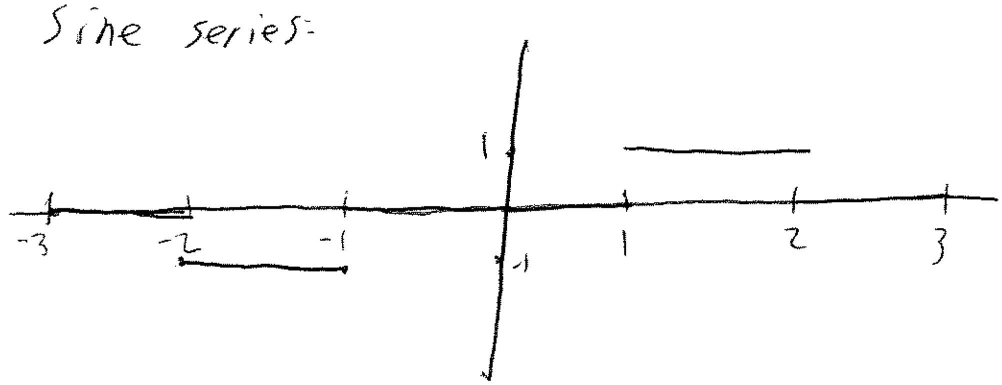
$$f(t) = \frac{2}{\pi}\left(\sin\left(\frac{\pi t}{3}\right) - \frac{2}{3}\sin\left(\frac{3\pi t}{3}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{3}\right) + \cdots \right.$$

Now the graphs are:

Cosine series:



Sine series:



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$$f(t) = t - t^2 \qquad 0 < t < 1$$

Cosine series:

$$a_0 = \frac{2}{1}\int_0^1 (t-t^2)\,dt$$

$$= 2\left(\frac{t^2}{2} - \frac{t^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

for $n \geq 1$

$$a_n = 2\int_0^1 (t - t^2)\cos(n\pi t)\,dt$$

$$= 2\left[\frac{t\sin(n\pi t)}{n\pi} + \frac{\cos(n\pi t)}{n^2\pi^2} - \frac{t^2\sin(n\pi t)}{n\pi} - \frac{2t\cos(n\pi t)}{n^2\pi^2} + \frac{2\sin(n\pi t)}{n^3\pi^3}\right]\Big|_0^1$$

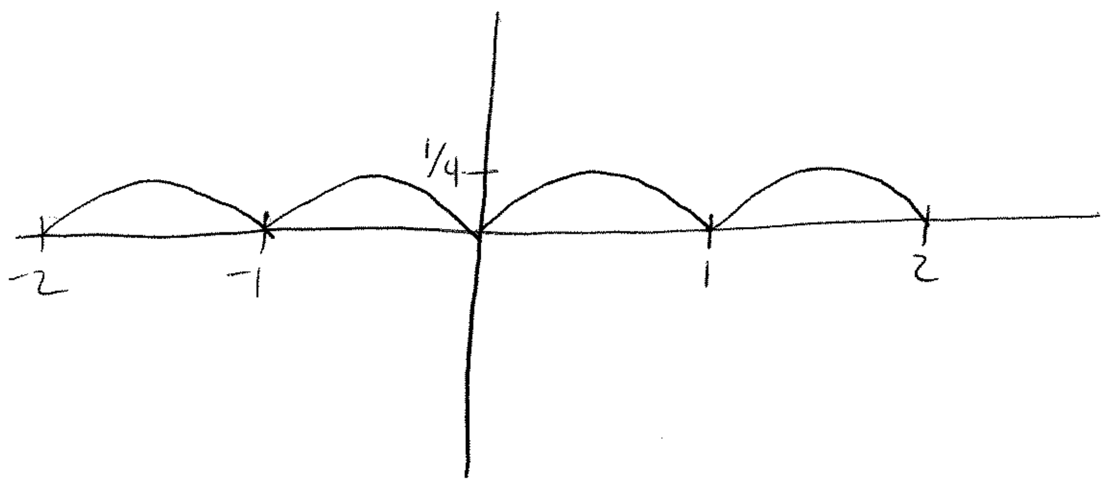
$$= 2\left[\frac{(-1)^n}{n^2\pi^2} - \frac{2(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$= 2\left[-\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] = \begin{cases} -\dfrac{4}{n^2\pi^2} & \text{for } n \text{ is even} \\ 0 & \text{for } n \text{ is odd.} \end{cases}$$

So, the cosine series is:

$$f(t) = \frac{1}{6} \cancel{\ast} - \frac{4}{\pi^2}\left(\frac{\cos(2\pi t)}{4} + \frac{\cos(4\pi t)}{16} + \frac{\cos(6\pi t)}{36} + \cdots\right)$$

the graph will be:



The sine series will be:

$$b_n = \frac{2}{1}\int_0^1 (t-t^2)\sin\left(\frac{n\pi t}{1}\right)dt$$

$$= 2\left[\frac{-t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{n^2\pi^2} + \frac{t^2\cos(n\pi t)}{n\pi} - \frac{2t\sin(n\pi t)}{n^2\pi^2} - \frac{2\cos(n\pi t)}{n^3\pi^3}\right]\Bigg|_0^1$$

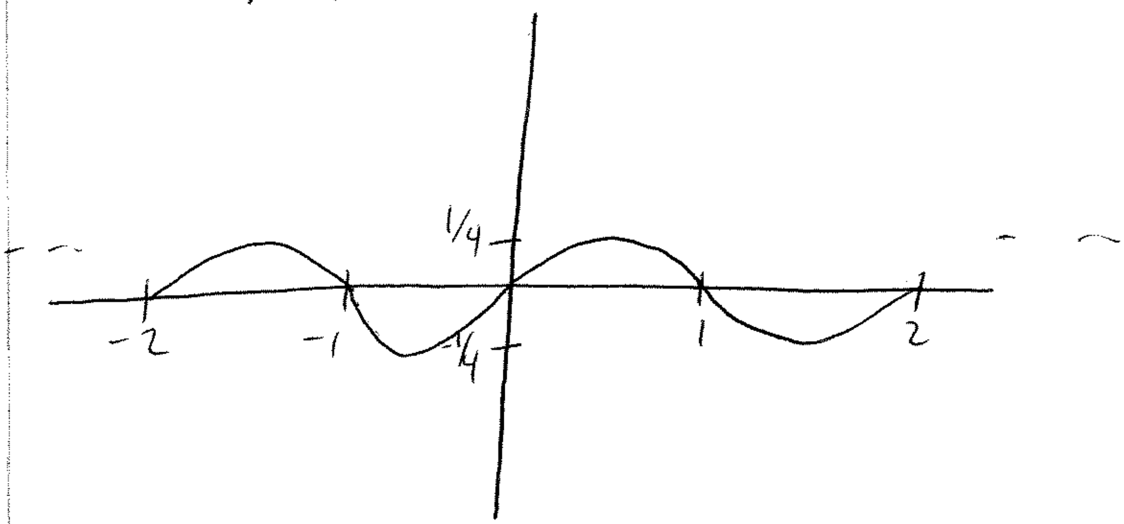
$$= 2\left(\left[-\frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{2(-1)^n}{n^3\pi^3}\right] - \left(-\frac{2}{n^3\pi^3}\right)\right)$$

$= + 8/n^3\pi^3$ for n ~~even~~ odd $\quad$ 0 for n ~~odd~~ even

So, the sine series will be:

$$+\frac{8}{\pi^3}\left(\frac{\sin(\pi t)}{1}+\frac{\sin(3\pi t)}{3^3}+\frac{\sin(5\pi t)}{5^3}+\cdots\right)$$

with graph:



9.3.13 Find the Fourier series solution to the ~~endpoint~~ endpoint value problem:

$$x'' + x = t, \qquad x(0) = x(1) = 0.$$

We note that $\sin(n\pi t)$ has value 0 at $t=0$ and $t=1$ for all n, so we'll want to use the sine series.

$$x(t) = \sum_{n=1}^{\infty} a_n \sin(n\pi t)$$

$$x'(t) = \sum_{n=1}^{\infty} a_n n\pi \cos(n\pi t)$$

$$x''(t) = \sum_{n=1}^{\infty} a_n (-n^2\pi^2) \sin(n\pi t)$$

and the sine series for $f(t) = t$ is:

$$b_n = 2\int_0^1 t \sin(n\pi t)\,dt$$

$$= 2\left[-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{n^2\pi^2} \right]\Bigg|_0^1$$

$$= \frac{-2(-1)^n}{n\pi}$$

So, we get the relation:

$$a_n \left(1 - n^2\pi^2\right) = \frac{2(-1)^{n+1}}{n\pi}$$

$$\Rightarrow \quad a_n = \frac{2(-1)^{n+1}}{\pi n\left(1-n^2\pi^2\right)}$$

So,

$$\boxed{x(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi t)}{n(n^2\pi^2-1)}.}$$

9.3.20. Substitute $t = \pi/2$ and $t = \pi$ in the series of Problem 19 to obtain the summations

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$

and

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$$

the series in problem 19 is:

$$\frac{1}{24}t^4 = \frac{\pi^2 t^2}{12} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\cos(nt) + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

For $t = \pi/2$ we get:

$$\frac{1}{24}\left(\frac{\pi^4}{16}\right) = \frac{\pi^2}{12}\left(\frac{\pi^2}{4}\right) - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\cos\left(\frac{n\pi}{2}\right) + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

Now,

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & \text{for } n = 4k \\ -1 & \text{for } n = 4k+2 \\ 0 & \text{for } n = 4k+1 \text{ or } 4k+3 \end{cases}$$

So, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\cos\left(\frac{n\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{(2n)^4}(-1)^n = \frac{1}{16}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

and so our series is:

$$\frac{\pi^4}{384} = \frac{\pi^4}{48} + \frac{15}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

$$\Rightarrow \quad \frac{-7\pi^4}{384} = \frac{15}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = -\frac{\pi^4}{720}$$

$$\Rightarrow \quad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}}$$

So, that's one of our identities.

For $t = \pi$ we get:

$$\frac{\pi^4}{24} = \frac{\pi^4}{12} - 2\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi)}{n^4} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

Now, $\cos(n\pi) = (-1)^n$. So, we get:

$$2\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{12} - \frac{\pi^4}{24} + 2\left(-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{\cancel{48}24} - \frac{\pi^4}{360} = \frac{15\pi^4 - \cancel{1}\pi^4}{\cancel{720}360} = \frac{14\pi^4}{\cancel{720}360} =$$

$$\Rightarrow 2\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{24} - \frac{14\pi^4}{720}$$

$$\Rightarrow 2\sum_{n=1}^{\infty}\frac{1}{n^4} = \cancel{\frac{16\pi^4}{720}} \quad \frac{16\pi^4}{720}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}}$$

That's the other. Now,

$$\sum_{n=1}^{\infty}\frac{1}{n^4} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$$

$$= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{90} + \frac{7\pi^4}{720}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{15\pi^4}{1440} = \frac{\pi^4}{96}.$$

So,

$$\boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}}$$

which is our last identity.