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Math 2280
Solutions

Assignment #12

8.2.1. Find a power series solution to the ODE:

$$(x^2-1)y'' + 4xy + 2y = 0$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

If we plug in these relations to the above ODE we get:

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n - \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + 4\sum_{n=0}^{\infty} n c_n x^n + 2\sum_{n=0}^{\infty} c_n x^n$$

~~We note here that for the lowest order term~~
So, we get the recurrence relation:

$$\sum_{n=0}^{\infty} \left[(n(n-1) + 4n + 2) c_n - (n+2)(n+1) c_{n+2} \right] x^n = 0$$

So,

$$c_{n+2} = \frac{(n^2+3n+2)}{(n+2)(n+1)} c_n = c_n.$$

So, we specify c_0 and c_1 "arbitrarily" and the rest of the coefficients are given by:

$$c_{n+2} = c_n.$$

So, our solutions are:

$$y_1(x) = c_0 \sum_{n=0}^{\infty} x^{2n}$$

and

$$y_2(x) = c_1 \sum_{n=1}^{\infty} x^{2n+1}$$

We see that this is just a modification of the geometric series, so the guaranteed radius of convergence is $\rho = 1$, so ~~sou~~ ~~s~~ our solution is well defined for $|x| < 1$.

Note for $|x| < 1$

$$\left.\begin{aligned} y_1(x) &= c_0 \sum_{n=0}^{\infty} x^{2n} = \frac{c_0}{1-x^2} \\ y_2(x) &= c_1 x \sum_{n=0}^{\infty} x^{2n} = \frac{c_1 x}{1-x^2}. \end{aligned}\right\} \text{Geometric series formula.}$$

So, our final solution can be written as:

$$\boxed{y(x) = \frac{c_0 + c_1 x}{1-x^2}}$$

8.2.7. $(x^2+3)y'' - 7xy' + 16y = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

Plugging these into the ODE we get:

$$\sum_{n=0}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} 3n(n-1) c_n x^{n-2}$$

$$- \sum_{n=0}^{\infty} 7n c_n x^n + \sum_{n=0}^{\infty} 16 c_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n^2 - 8n + 16) c_n + 3(n+2)(n+1) c_{n+2} \right] x^n$$

So, we get the recurrence relation

$$c_{n+2} = \frac{-(n-4)^2 c_n}{3(n+2)(n+1)}$$

Now, we can see that we can specify c_0 and c_1 "arbitrarily" and the rest of the coefficients are determined by the above recurrence relation.

For the ~~even terms we get:~~

$\cancel{c_0 = c_0 \quad c_2 = \frac{-4c_0}{6} = -\frac{2}{3}c_0 \quad c_4 = 0}$

~~and all higher terms are 0.~~

$$c_{n+2} = \frac{-(n-4)^2 c_n}{3(n+2)(n+1)}$$

Now, for the odd terms we get:

$$c_1 = c_1 \qquad c_3 = \frac{-9c_1}{3\cdot(3\cdot 2)} \qquad c_5 = \frac{-c_3}{3\cdot(5\cdot 4)} = \frac{9c_1}{3^2(5\cdot 4\cdot 3\cdot 2\cdot 1)}$$

$$c_7 = \frac{-c_5}{3\cdot(7\cdot 6)} = \frac{-9c_1}{3^3(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}$$

$$c_9 = \frac{-9c_7}{3(9\cdot 8)} = \frac{81c_1}{3^4(9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}$$

and in general for $n \geq 3$

$$c_{2n+1} = \frac{(-1)^n[(2n-5)!!]^2 c_1}{3^{n-2}(2n+1)!}$$

Now, the even terms are much more simple:

$$c_0 = c_0 \qquad c_2 = \frac{-16c_0}{3(2\cdot 1)} = -\frac{8}{3}c_0$$

$$c_4 = \frac{-4c_2}{3(4\cdot 3)} = \frac{32}{3^3\cdot 4} = \frac{8}{27}c_0$$

$c_6 = 0$ and all higher terms are 0.

So, our final solution is:

$$y(x) = c_0\left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right)$$

$$+ c_1\left(x - \frac{x^3}{2} + \frac{x^5}{120} + \sum_{n=3}^{\infty} \frac{(-1)^n\left[(2n-5)!!\right]^2}{3^{n-2}(2n+1)!} x^{2n+1}\right)$$

Now, the radius of convergence is:

$$\lim_{n\to\infty} \left| \frac{\frac{\left[(2n-5)!!\right]^2}{3^{n-2}(2n+1)!}}{\frac{\left[(2n-3)!!\right]^2}{3^{n-1}(2n+2)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3(2n+2)}{(2n-3)^2} \right| = 0.$$

So, the solution is useless

Note: The solution in the back of the book is wrong. It should have the $(2n-5)!!$ squared. Yet <u>another</u> typo.

8.2.14. $y'' + xy = 0$ (an Airy equation)

$$y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

$$y = \sum_{n=0}^{\infty} n(n-1) c_n x^n$$

Plugging these in we get:

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

Now, we first note that the x^0 term, the constant term, just shows up in the first sum, and so we get $2c_2 = 0$, which requires $c_2 = 0$.

Now, c_0 and c_1 will be arbitrary, and we get the recurrence relation:

$$\sum_{n=0}^{\infty} \left[(n+3)(n+2) c_{n+3} x^{n+1} + c_n x^{n+1}\right] = 0.$$

So, $$c_{n+3} = \frac{-c_n}{(n+3)(n+2)}$$

The $3n$ terms are:

$c_0 = c_0$ $\quad c_3 = \frac{-c_0}{3 \cdot 2}$ $\quad c_6 = \frac{-c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}$

$$c_9 = \frac{-c_6}{9 \cdot 8} = \frac{-c_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} = \frac{-(7 \cdot 4) c_0}{9!}$$

and in general:

$$c_{3n} = \frac{(-1)^n (1 \cdot 4 \cdot 7 \cdots (3n-2))\, c_0}{(3n)!}$$

Similarly,

$$c_1 = c_1 \qquad c_4 = \frac{-c_1}{4 \cdot 3} \qquad c_7 = \frac{-c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}$$

$$c_{10} = \frac{-c_7}{10 \cdot 9} = \frac{-c_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} = \frac{-(2 \cdot 5 \cdot 8)\, c_1}{10!}$$

and in general

$$c_{3n+1} = \frac{(-1)^n (2 \cdot 5 \cdot 8 \cdots (3n-2)}{(3n+1)!}$$

And all c_{3n+2} terms are 0. So, the final solution is:

$$\boxed{\begin{aligned} y(x) = {} & c_0 \sum_{n=0}^{\infty} \frac{(-1)^n (1 \cdot 4 \cdot 7 \cdots (3n-2))}{(3n)!} x^{3n} \\ & + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n (2 \cdot 5 \cdot 8 \cdots (3n-2)}{(3n+1)!} x^{3n+1} \end{aligned}}$$

The radius of convergence will be:

$$\lim_{n \to \infty} \left| \frac{\frac{(1 \cdot 4 \cdot 7 \cdots (3n-2)}{(3n)!}}{\frac{(1 \cdot 4 \cdot 7 \cdots (3n+1)}{(3n+1)!}} \right| = \lim_{n \to \infty} \left| \frac{3n+1}{3n+1} \right| = 1.$$

So, the solution works for $|x|<1$.

8.2.16 $(1+x^2)y'' + 2xy' - 2y = 0$; $y(0)=0$, $y'(0)=1$.

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

Plugging these in we get:

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} [n(n-1)c_n + 2nc_n - 2c_n] x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} (n^2 + n \overset{-2}{\cancel{-4}}) c_n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + (n+2)(n-1) c_n] x^n$$

$$\Rightarrow \quad c_{n+2} = \frac{-(n-1)c_n}{(n+1)}$$

So, c_0 and c_1 ~~can b~~ are determined by the initial conditions, while the rest of the coefficients are determined by the above recurrence relation.

$$c_{n+2} = \frac{-(n-1)c_n}{(n+1)}$$

The even terms:

$$c_0 = c_0 \quad c_2 = c_0 \quad c_4 = \frac{-c_2}{3} = \frac{-c_0}{3}$$

$$c_6 = \frac{-3c_4}{5} = \frac{3c_0}{5 \cdot 3} = \frac{c_0}{5}$$

$$c_8 = \frac{-5c_6}{7} = \frac{-5c_0}{7 \cdot 5} = \frac{-c_0}{7}$$

In general

$$c_{2n} = \frac{(-1)^{n+1} c_0}{(2n-1)} \quad \text{for } n \geq 1.$$

For the odd terms:

$$c_1 = c_1 \quad c_3 = 0 \quad c_5 = 0 \text{ etc----.}$$

So, our final solution is:

$$y(x) = c_0\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)} x^{2n}\right) + c_1 x$$

Now, using our initial conditions we get:

$$y(0) = c_0 = 0.$$

So, $y(x) = c_1 x.$ $\quad y'(x) = c_1 = 1.$

So, $\boxed{y(x) = x}$

I'm an idiot and did the wrong problem.
So, you get a bonus solution to 8.2.16.

8.2.17. $y'' + xy' - 2y = 0; \quad y(0) = 1, \; y'(0) = 0.$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

So,

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 2 c_n x^n = 0.$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} + (n-2) c_n\right] x^n = 0.$$

This gives us the recursion relation:

$$c_{n+2} = \frac{-(n-2) c_n}{(n+2)(n+1)}$$

So, c_0 and c_1 are determined by the initial conditions, and the rest of the coefficients are determined by c_0 and c_1.

Even terms:

$$c_0 = c_0 \qquad c_2 = \frac{+2c_0}{2 \cdot 1} = +c_0 \qquad c_4 = 0$$

all other even terms are 0.

Odd terms:

$$c_1 = c_1 \qquad c_3 = \frac{+c_1}{3 \cdot 2} \qquad c_5 = \frac{-c_3}{5 \cdot 4} = \frac{-c_1}{5 \cdot 4 \cdot 3 \cdot 2}$$

$$c_7 = \frac{-3c_5}{7 \cdot 6} = \frac{3c_1}{7!}$$

and in general we have

$$\cancel{c_1 \left(1 + \frac{1}{3!} - \frac{1}{5!} \right.}$$

c_1, $\frac{c_1}{3!}$, and for the rest

$$c_{2n+1} = \frac{(-1)^{n+1}(2n-3)!!}{(2n+1)!} x^{2n+1}.$$

So, our solution is:

$$y(x) = c_0 \left(1 + x^2 \right) + c_1 \left(x + \frac{x^3}{3!} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2n-3)!!}{(2n+1)!} x^{2n+1} \right)$$

Now,

$$y(0) = c_0 = 1$$

and

$$y'(0) = c_1 = 0.$$

So,

$$\boxed{y(x) = 1 + x^2}$$

8.2.32 Follow the steps outlined in this problem to establish Rodrigues's formula:

$$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n} (x^2-1)^n$$

for the ~~n-th~~ nth-degree Legendre polynomial.

a) Show that $v = (x^2-1)^n$ satisfies the differential equation

$$(1-x^2)v' + 2nxv = 0$$

Differentiate each side of this equation to obtain

$$(1-x^2)v'' + 2(n-1)xv' + 2nv = 0.$$

Well,

$$v' = \frac{d}{dx}(x^2-1)^n = n(x^2-1)^{n-1} 2x.$$

So

$$(1-x^2)v' + 2nxv = (1-x^2)n(x^2-1)^{n-1}2x + 2nx(x^2-1)^n$$

$$= -2nx(x^2-1)^n + 2nx(x^2-1)^n = 0.$$

So, it satisfies the ODE.

If we differentiate the ODE we get:

$$(1-x^2)v'' + (-2x)v' + 2nxv' + 2nv = 0$$

$$= \boxed{(1-x^2)v'' + 2(n-1)xv' + 2nv = 0}$$

b) Differentiate each side of the last equation n times in succession to obtain

$$(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0.$$

Thus $u = v^{(n)} = D^n(x^2-1)^n$ satisfies Legendre's equation of order n.

This can be proven by induction.

Base case: $n = 0$

In this case we just plug $n=0$ into our earlier equation to obtain

$$(1-x^2)v'' - 2xv' = 0$$

which is what we want.

Induction: Suppose it's true for up to $n-1$.

Actually, this is not how I want to do it.

I will instead prove by induction that for $k \le n$ the result of differentiating

$$(1-x^2)v'' + 2(n-1)xv' + 2nv = 0$$

k times is:

$$0 = (1-x^2)v^{(k+2)} + 2(n-(k+1))xv^{(k+1)} + \sum_{m=0}^{k} 2(n-m)v^{(k}$$

Base case $k=0$:

We get:

$$(1-x^2)v'' + 2(n-1)xv' + 2nv = 0$$

which is our original equation, so it checks out.

Now, suppose it's true for up to $k-1$.

Then, if we differentiate

$$(1-x^2)v^{(k+1)} + 2(n-k)xv^{(k)} + \sum_{n=0}^{k-1} 2(n-m)v^{(k-1)}$$

we get:

$$(1-x^2)v^{(k+2)} - 2xv^{(k+1)} + 2(n-k)xv^{(k+1)}$$

$$+2(n-k)v^{(k)} + \sum_{n=0}^{k-1} 2(n-m)v^{(k)}$$

$$= (1-x^2)v^{(k+2)} + 2(n-(k+1))xv^{(k+1)} + \sum_{m=0}^{k} 2(n-m)v^{(k)}$$

So, the formula works

If we plug in $k=n$ we get:

$$(1-x^2)v^{(n+2)}+2(n-(n+1))xv^{(n+1)}+\sum_{m=0}^{n}2(n-m)v^{(n)}$$

$$=(1-x^2)v^{(n+2)}-2xv^{(n+1)}+\left[2n\sum_{m=0}^{n}1-2\sum_{m=0}^{n}m\right]v^{(n)}$$

$$=(1-x^2)v^{(n+2)}-2xv^{(n+1)}+\left(2n(n+1)-2\left(\frac{n^2+n}{2}\right)\right)v^{(n)}$$

$$=(1-x^2)v^{(n+2)}-2xv^{(n+1)}+(n^2+n)v^{(n)}$$

$$=\boxed{(1-x^2)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}}$$

So, it works! Whoa! That was hard!

c) Show that the coefficient of x^n in u is $(2n)!/n!$; then state why this proves Rodriguez's formula.

$$u=D^n(x^2-1)^n$$

$$=\frac{d^n}{dx^n}\left(x^{2n}+\text{lower order terms}\ldots\right)$$

$$=2n(2n-1)(2n-2)\cdots(2n-(n-1))x^n+\text{lower order terms}$$

$$=\frac{(2n)!}{n!}x^n+\text{lower order terms.}$$

So,

$$\cancel{P_n(x)} = \frac{u}{n!\,2^n}$$

satisfies Legendre's equation of order n. It is a polynomial, and as explained in the textbook there is only one polynomial that satisfies Legendre's equation of order n, which is:

because $u = v^{(n)}$ does.

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k (2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!} x^{n-2k}.$$

~~The highest~~

So, it must be $k P_n(x) = \frac{u}{n!\,2^n}$

where k is a constant.

The highest order term in $P_n(x)$ is when $k=0$ and is

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n + \text{lower order terms}$$

Now, $\frac{u}{n!\,2^n} = \frac{(2n)!}{2^n (n!)^2} x^n + \text{lower order terms}$

So, $k=1$ and indeed

$$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n} (x^2-1)^n.$$

8.3.1. Determine whether $x=0$ is an ordinary point, a regular singular point, or an irregular singular point, find the exponents of the differential equation at $x=0$.

$$xy'' + (x - x^3)y' + (\sin x)y = 0.$$

$$\Rightarrow \quad y'' + (1-x^2)y' + \left(\frac{\sin x}{x}\right)y = 0.$$

Now, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$,

so it's analytic at $x=0$.

So, $x=0$ is an ordinary point.

8.3.8. $(6x^2+2x^3)y'' + 21xy' + 9(x^2-1)y = 0$

$$\Rightarrow \quad y'' + \frac{21xy'}{2x^2(3+x)} + \frac{9(x+1)(x-1)}{2x^2(x+3)}y = 0$$

$$= y'' + \left(\frac{21}{2x(3+x)}\right)y' + \left(\frac{9(x+1)(x-1)}{2x^2(x+3)}\right)y = 0.$$

Now, $\lim_{x\to 0}\left(\frac{21}{2x(x+3)}\right)$ is undefined $(\pm\infty)$

so $x=0$ is not an ordinary point.

However,

$$p(x) = xP(x) = \frac{21}{2(x+3)} \qquad q(x) = x^2Q(x) = \frac{9(x+1)(x-1)}{2(x+3)}$$

have defined limits as $x \to 0$, so are analytic at $x=0$.

So, $x=0$ is a regular singular point.

8.3.15 If $x=a\neq 0$ is a singular point of a second-order linear differential equation, then the substitution $t=x-a$ transforms it into a differential equation having $t=0$ as a singular point. We then attribute the the original equation at $x=a$ the behavior of the new equation at $t=0$. Classify the singular points.

$$(x-2)^2 y'' - (x^2-4)y' + (x+2)y = 0$$

$$\Rightarrow \quad y'' - \frac{(x+2)}{(x-2)} y' + \frac{(x+2)}{(x-2)^2} y = 0$$

which has a singular point (denominator is 0) ~~when num~~ at $x=2$. So, substituting $t=x-2$ we get:

$$y'' - \frac{(t+4)}{t} y' + \frac{(t+4)}{t^2} y = 0$$

$$p(t) = t\left(\frac{t+4}{t}\right) = t+4$$

$$q(t) = t^2\left(\frac{t+4}{t^2}\right) = t+4.$$

Both are analytic at $t=0$. So, $t=0$ $(x=2)$ is a regular singular point.

8.3.18 Find two linearly independent Frobenius series solutions of the ODE.

$$2xy'' + 3y' - y = 0$$

$$\Rightarrow y'' + \frac{3}{2x}y' - \frac{1}{2x}y = 0$$

as $x \to 0$ we have $P(x)$ and $Q(x)$ with undefined limits, so $x=0$ is not an ordinary point.

$$p(x) = xP(x) = \tfrac{3}{2}$$
$$q(x) = x^2Q(x) = -\tfrac{1}{2}x$$

are both analytic at $x=0$, so $x=0$ is a regular singular point.

$$p_0 = p(0) = 3/2 \qquad q_0 = q(0) = 0.$$

So, we get the indicial equation:

$$r(r-1) + \tfrac{3}{2}r = 0$$

$$\Rightarrow r^2 + \tfrac{1}{2}r = 0$$

Which has roots $\{0, -\tfrac{1}{2}\}$.

Our Frobenius series will be:

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r)c_n\, x^{n+r-1}$$
$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, x^{n+r-2}.$$

Plugging these relations into the ODE we get:

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + 3\sum_{n=0}^{\infty}(n+r)\, c_n\, x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[\, 2(n+r)(n+r-1) + 3(n+r)\right] c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+r+1)(2n+2r+3)\, c_{n+1} - c_n\right] x^{n+r} = 0.$$

So, we set up the indicial equation so that we could c_0 arbitrarily, and then we have the recurrence relation:

$$c_{n+1} = \frac{c_n}{(n+r+1)(2n+2r+3)}$$

For $r = -\frac{1}{2}$ we get:

$$c_{n+1} = \frac{c_n}{(n+\frac{1}{2})(2n+2)} = \frac{c_n}{(2n+1)(n+1)}$$

and in general

$$c_{n+1} = \frac{c_0}{(n+1)!\,(2n+1)!!}$$

For $r=0$ we get:

$$c_{n+1} = \frac{c_n}{(n+1)(2n+3)}$$

and in general

$$c_{n+1} = \frac{c_0}{(n+1)!\,(2n+3)!!}$$

So, our solution is:

$$\cancel{y(x) = x^{-1/2} a_0 \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!\,(2n+1)!!} + c_1}$$

$$\boxed{y(x) = a_0\, x^{-1/2} \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!\,(2n+1)!} + b_0 \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!\,(2n+3)!!} + b_0}$$

8.3.24. $3x^2y'' + 2xy' + x^2y = 0.$

$$\Rightarrow y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0.$$

$\lim_{x\to 0}\left(\frac{2}{3x}\right)$ is undefined, so $x=0$ is not an ordinary point.

$$p(x) = xP(x) = \frac{2}{3}$$
$$q(x) = x^2Q(x) = \frac{1}{3}x^2$$

are analytic at $x=0$ with

$$p_0 = p(0) = 2/3$$
$$q_0 = q(0) = 0.$$

So, our indicial equation is:

$$r(r-1) + \frac{2}{3}r + 0 = 0$$

$$\Rightarrow r\left(r-\frac{1}{3}\right) = 0 \Rightarrow r = \left\{0, \frac{1}{3}\right\}$$

Now, our Frobenius series will be:

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$
$$y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$
$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

Plugging this into our ODE we get:

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)\, c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0.$$

Now, the x^r ~~##~~ term lets us arbitrarily specify c_0 based on our choice of r.

The x^{r+1} term has coefficient:

$$[3(1+r)r + 2(1+r)]\, c_1$$

$$= (3r+2)(1+r)\, c_1$$

Now, $(3r+2)(1+r) \neq 0$ for $r = \{0, \frac{1}{3}\}$

and so $c_1 = 0$.

Now, for higher order terms we get the recursion relation:

$$[3(n+r+2)(n+r+1) + 2(n+r+2)]\, c_{n+2} + c_n = 0$$

$$\Rightarrow (n+r+2)(3n+3r+5)\, c_{n+2} + c_n = 0$$

$$\Rightarrow c_{n+2} = \frac{-c_n}{(n+r+2)(3n+3r+5)}$$

Now, all odd terms will be 0 as $c_1 = 0$.

For the even terms we get:

For $r = \frac{1}{3}$

$$\cancel{c_{n+2} = \frac{-c_n}{(n+\frac{2}{3})(3n+8)} = \frac{-3c_n}{(3n+7)(3n+8)}}$$

$$\cancel{c_0 = c_0}$$

$$\cancel{c_2 = \frac{-3c_0}{7\cdot 8}}$$

$$\cancel{c_4 = \frac{-3c_2}{13\cdot 14} = \frac{9c_0}{(13\cdot 7)\cdot(14\cdot 8)}}$$

$$c_{n+2} = \frac{-c_n}{(n+7/3)(3n+6)} = \frac{-c_n}{(3n+7)(n+2)}$$

So, $c_0 = c_0$

$$c_2 = \frac{-c_0}{7\cdot 2}$$

$$c_4 = \frac{-c_2}{\cancel{10\cdot 4}\ 13\cdot 4} = \cancel{\frac{c_0}{(7\cdot 10)(2\cdot 4)}}\ \frac{c_0}{(7\cdot 13)(2\cdot 4)}$$

$$c_6 = \frac{-c_4}{\cancel{13\cdot 6}\ 17\cdot 4} = \cancel{\frac{-c_0}{(7\cdot 10\cdot 13)(2\cdot 4\cdot 6)}} = \frac{-c_0}{(7\cdot 13\cdot 17)(2\cdot 4\cdot 6)}$$

and in general:

$$c_{2n} = \cancel{\frac{(-1)^n c_0}{2^n n!\,(7\cdot 10\cdot 13\cdots(3n+7))}} = \frac{(-1)^n c_0}{2^n n!\,(7\cdot 13\cdot 17\cdots(6n+1))},$$

For $r = 0$

$$c_{n+2} = \frac{-c_n}{(n+2)(3n+5)}$$

So, $$c_0 = c_0$$

$$c_2 = \frac{-c_0}{2\cdot 5}$$

$$c_4 = \frac{-c_2}{4\cdot \cancel{8}\,11} = \frac{c_0}{(2\cdot 4)(5\cdot 11)}$$

and in general:

$$\cancel{c_{2n} = \frac{(-1)^n c_0}{2^n n!\,(5\cdot 8\cdots}}$$

$$c_{2n} = \frac{(-1)^n c_0}{2^n n!\,(5\cdot 11\cdots(6n-1))}$$

So, our solution is:

$$\boxed{\begin{aligned} y(x) = {} & a_0 x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\,(7\cdot 13\cdots(6n+1))} \\ & + b_0 \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\,(5\cdot 11\cdots(6n-1))} + b_0 \end{aligned}}$$