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Math 2280
Solutions

Assignment #11

7.5.1. Find the inverse Laplace transform and sketch the graph of the inverse transform.

$$F(s) = \frac{e^{-3s}}{s^2}$$

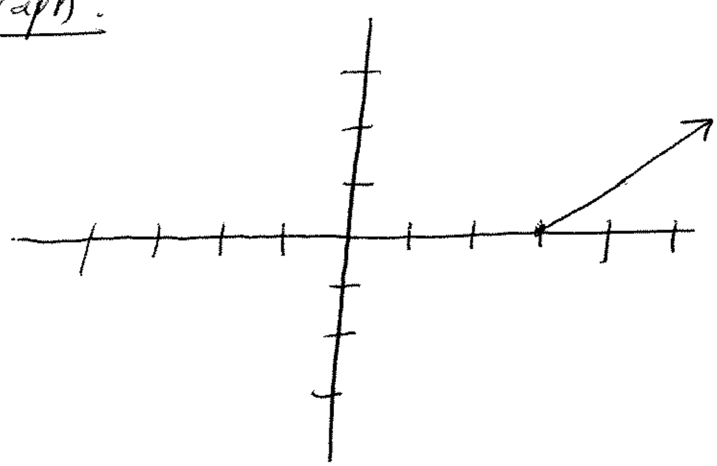
$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{e^{-3s}\frac{1}{s^2}\} = u(t-3)f(t-3)$$

where $f(t) = \mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$.

So,

$$f(t) = u(t-3)(t-3)$$

Graph:



7.5.6.

$$F(s) = \frac{s e^{-s}}{s^2 + \pi^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left(e^{-s}\left(\frac{s}{s^2+\pi^2}\right)\right)$$

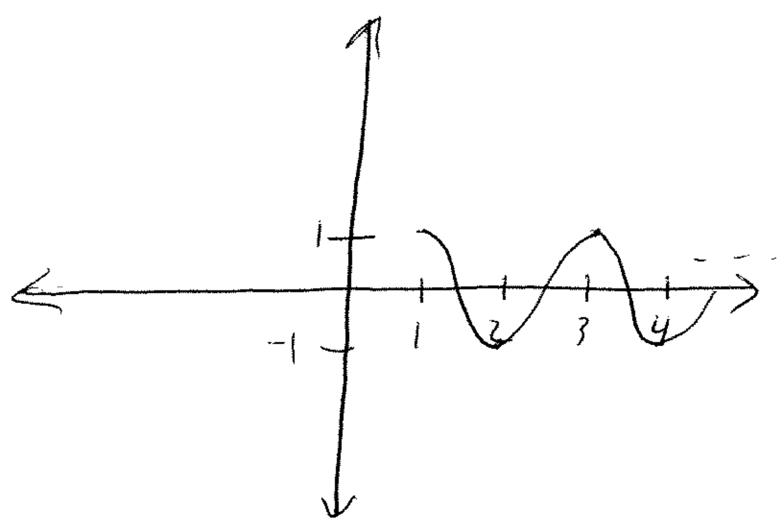
$$= u(t-1) f(t-1)$$

where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos(\pi t)$$

So,

$$\mathcal{L}^{-1}\{F(s)\} = u(t-1)\cos(\pi(t-1))$$



?.5.15 Find the Laplace transform of the function $f(t)$:

$$f(t) = \sin(t) \text{ if } 0 \leq t \leq 3\pi$$
$$f(t) = 0 \text{ if } t > 3\pi$$

$$= s(t)\,[u(t) - u(t-3\pi)]$$

$$= \cancel{}\ u(t)\sin(t) - u(t-3\pi)\sin(t)$$

Now,

$$u(t)\sin(t) - u(t-3\pi)\sin(t)$$

$$= u(t)\sin(t) + u(t-3\pi)\sin(\cancel{3\pi}\ t-3\pi)$$

$$\mathcal{L}\{u(t)\sin(t) + u(t-3\pi)\sin(t-3\pi)\}$$

$$= \frac{1}{s^2+1} + e^{-3\pi s}\left(\frac{1}{s^2+1}\right)$$

$$= \boxed{\frac{1+e^{-3\pi s}}{s^2+1}}$$

7.5.21. $f(t) = t$ if $t \le 1$
$f(t) = 2-t$ if $1 \le t \le 2$
$f(t) = 0$ if $t > 2$.

$$f(t) = t - u(t-1)t + u(t-1)(2-t) - u(t-2)(2-t)$$

$$= t - u(t-1)(t-1) - u(t-1) + 2u(t-1) - u(t-1)(t-1)$$
$$- u(t-1) - 2u(t-2) + u(t-2)(t-2) + 2u(t-2)$$

$$= t - u(t-1)(t-1) - u(t-1) + 2u(t-1) - u(t-1)(t-1)$$
$$- u(t-1) + u(t-2)(t-2)$$

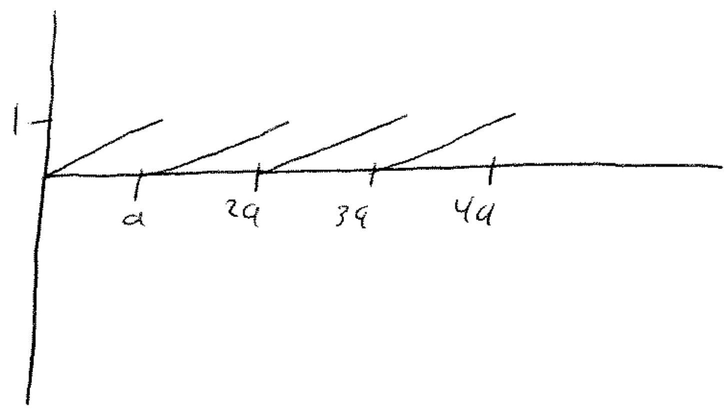
$$= t - 2u(t-1)(t-1) + u(t-2)(t-2)$$

$$\mathcal{L}\{t - 2u(t-1)(t-1) + u(t-2)(t-2)\}$$

$$= \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

$$= \frac{1-2e^{-s}+e^{-2s}}{s^2} = \boxed{\frac{(1-e^{-s})^2}{s^2}}$$

7.5.26 Apply Theorem 2 to show that the Laplace transform of the sawtooth function f(t):



is $F(s) = \dfrac{1}{as^2} - \dfrac{e^{-as}}{s(1-e^{-as})}$

The period here is a, and we have, according to Theorem 2:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-as}} \int_0^a \frac{e^{-st}t}{a}\,dt$$

$$\int_0^a \frac{e^{-st}t}{a}\,dt \qquad u = t/a \quad du = \tfrac{1}{a}dt \qquad dv = e^{-st} \quad v = -e^{-st}/s$$

$$= -\frac{te^{-st}}{as}\Big|_0^a + \frac{1}{as}\int_0^a e^{-st}\,dt$$

$$= \frac{-ae^{-as}}{as} + \frac{1}{as}\left(\frac{-e^{-st}}{s}\right)\Big|_0^a$$

$$= \frac{-ae^{-as}}{as} - \frac{e^{-as}}{as^2} + \frac{1}{as^2} = \frac{1}{as^2} - \frac{(1+as)e^{-as}}{as^2}$$

So,

$$\mathcal{L}\{f(t)\} = \left(\frac{1}{1-e^{-as}}\right)\left(\frac{1}{as^2} - \frac{(1+as)e^{-as}}{as^2}\right)$$

$$= \left(\frac{1}{1-e^{-as}}\right)\left(\frac{1-e^{-as}}{as^2}\right) - \frac{as\,e^{-as}}{as^2(1-e^{-as})}$$

$$= \frac{1}{as^2} - \frac{a\,e^{-as}}{as(1-e^{-as})}$$

$$= \boxed{\frac{1}{as^2} - \frac{e^{-as}}{s(1-e^{-as})}}$$

7.6.1. Solve the initial value problems:

$$x'' + 4x = \delta(t); \quad x(0) = x'(0) = 0$$

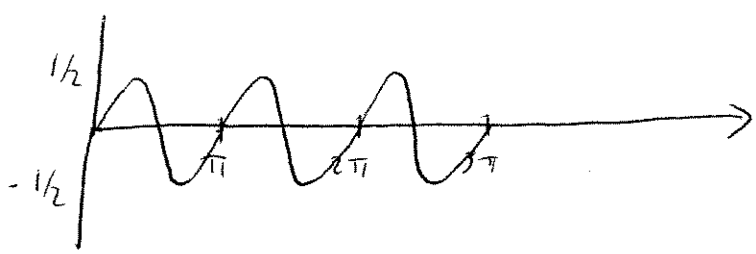
Taking the Laplace transform of both sides:

$$s^2 X(s) + 4X(s) = 1$$

$$\Rightarrow \quad X(s) = \frac{1}{s^2+4}$$

$$\Rightarrow \boxed{x(t) = \tfrac{1}{2}\sin(2t)}$$

Graph:



7.6.6 Solve the initial value problem:

$$x'' + 9x = \delta(t - 3\pi) + \cos(3t); \quad x(0) = x'(0) = 0.$$

Taking the Laplace transform:

$$s^2 X(s) + 9X(s) = e^{-3\pi s} + \frac{s}{s^2+9}$$

$$X(s) = \frac{e^{-3\pi s}}{s^2+9} + \frac{s}{(s^2+9)^2}$$

The inverse Laplace transform of this is:

$$\mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{s^2+9}\right) = u(t-3\pi)\, f(t-3\pi)$$

where

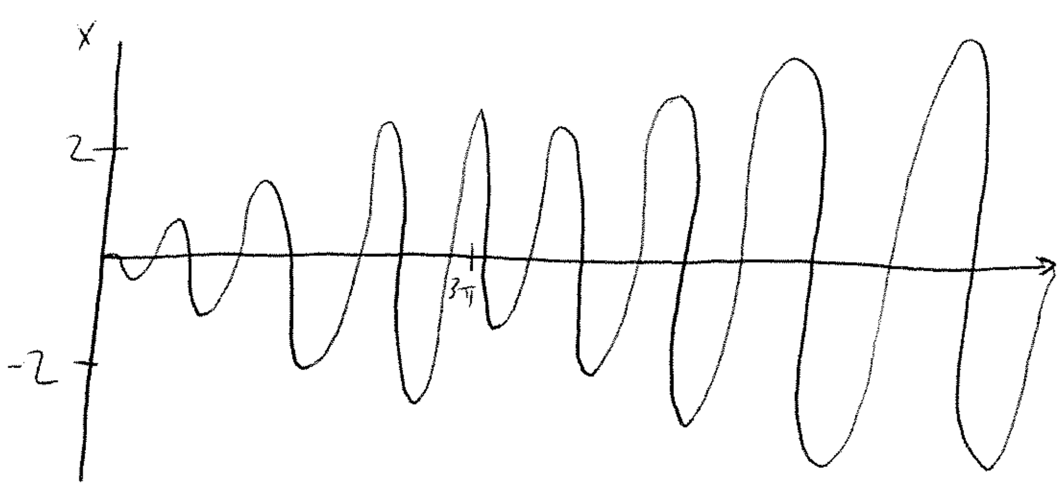
$$f(t) = \mathcal{L}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$$

$$f(t-3\pi) = \frac{1}{3}\sin(3(t-3\pi)) = -\frac{1}{3}\sin(3t)$$

So,
$$\mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{s^2+9}\right) = -\frac{1}{3}u(t-3\pi)\sin(3t)$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+9)^2}\right) = \frac{1}{6}t\sin(3t)$$

$$\Rightarrow \boxed{x(t) = \frac{t\sin(3t) - 2u(t-3\pi)\sin(3t)}{6}}$$



7.6.11

~~X~~ Apply Duhamel's principle to write an integral formula for the solution of each initial value problem.

$$x'' + 6x' + 8x = f(t) \qquad x(0) = x'(0) = 0.$$

$$\Rightarrow \quad s^2 X(s) + 6s X(s) + 8X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2 + 6s + 8} = W(s) F(s)$$

where $$W(s) = \frac{1}{s^2+6s+8} = \frac{1}{(s+3)^2 - 1}$$

$$\cancel{= \frac{e^{-3t}}{}}$$

$$\mathcal{L}^{-1}\{W(s)\} = e^{-3t}\sinh(t)$$

And so,

$$x(t) = \int_0^t e^{-3\tau}\sinh(\tau) f(t-\tau)\, d\tau$$

The convolution of ~~$\mathcal{L}^{-1}\{W(s$~~ $\mathcal{L}^{-1}(W(s))$ and $f(t)$.

7.6.14. Verify that $u'(t-a) = \delta(t-a)$ by solving the problem

$$x' = \delta(t-a); \quad x(0) = 0$$

to obtain $x(t) = u(t-a)$.

Taking the Laplace transform of both sides:

$$s\,X(s) = e^{-as}$$

$$X(s) = \frac{e^{-as}}{s}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{e^{-as}\left(\tfrac{1}{s}\right)\}$$

$$= u(t-a)\,f(t-a)$$

$$f(t) = \mathcal{L}^{-1}\left(\tfrac{1}{s}\right) = 1.$$

So, $\boxed{x(t) = u(t-a)}$

7.6.15. This problem deals with a mass m on a spring (with constant k) that receives an impulse $p_0 = m v_0$ at time $t = 0$.

Show that the initial value problems

$$m x'' + kx = 0; \quad x(0) = 0, \quad x'(0) = v_0$$

and

$$m x'' + kx = p_0\, \delta(t); \quad x(0) = 0, \quad x'(0) = 0$$

have the same solution. Thus the effect of $p_0\, \delta(t)$ is, indeed, to impart to the particle an initial momentum p_0.

The first problem has the solution:

$$c_1 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

$x(0) = c_1 = 0$. So, $c_1 = 0$.

$$x(t) = c_2 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

$$x'(t) = c_2 \sqrt{\tfrac{k}{m}} \cos\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

$$x'(0) = c_2 \sqrt{\tfrac{k}{m}} = v_0 \Rightarrow c_2 = v_0 \sqrt{\tfrac{m}{k}}.$$

So,

$$\boxed{x(t) = v_0 \sqrt{\tfrac{m}{k}} \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)}$$

On the other hand

$m x'' + kx = p_0 \, \delta(t) \, ; \quad x(0) = 0 \, , \quad x'(0) = 0.$

taking the Laplace transform we get:

$m s^2 X(s) + k X(s) = p_0$

$$\Rightarrow \quad X(s) = \frac{p_0}{ms^2 + k} = \frac{p_0/m}{s^2 + k/m}$$

$$= \frac{p_0}{m} \sqrt{\frac{m}{k}} \left(\frac{\sqrt{k/m}}{s^2 + k/m} \right) \qquad p_0 = m v_0$$

$$= v_0 \sqrt{\frac{m}{k}} \left(\frac{\sqrt{k/m}}{s^2 + k/m} \right)$$

$$\mathcal{L}^{-1}(X(s)) = \boxed{v_0 \sqrt{\frac{m}{k}} \sin \left(\sqrt{\tfrac{k}{m}} \, t \right)}$$

So, the ODEs have the same solution.

3.1.2. Find a power series solution of the given ODE. Determine the radius of convergence, and use it to identify the series solution in terms of familiar functions.

$$y' = 4y$$

$$\Rightarrow y' - 4y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$$

$$\sum_{n=0}^{\infty} c_n n x^{n-1} - 4\sum_{n=0}^{\infty} c_n x^n = 0$$

c_0 is arbitrary.

$$= \sum_{n=0}^{\infty} [c_{n+1}(n+1) - 4c_n] x^n = 0$$

So,

$$c_{n+1}(n+1) - 4c_n = 0$$

$$\Rightarrow c_{n+1} = \frac{4c_n}{n+1}$$

$$c_0 = c_0 \qquad c_2 = \frac{4c_1}{2} = \frac{4^2 c_0}{2 \cdot 1}$$

$$c_1 = \frac{4c_0}{1} \qquad c_3 = \frac{4c_2}{3} = \frac{4^3 c_0}{3 \cdot 2 \cdot 1}$$

In general,

$$c_n = \frac{c_0 4^n}{n!}$$

The radius of convergence will be:

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{c_0 4^n / n!}{c_0 4^{n+1} / (n+1)!} \right| = \lim_{n\to\infty} \frac{n+1}{4} = \infty$$

So, it converges for all x.

The series solution will be:

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{4^n x^n}{n!}$$

$$= c_0 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$= \boxed{c_0 e^{2x}}$$

8.1.8.

$$2(x+1)y' = y$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} c_n n x^{n-1}$$

So, we get:

$$2\sum_{n=0}^{\infty} c_n n x^n + 2\sum_{n=0}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_n x^n$$

c_0 is arbitrary.

We then get the recurrence relation:

$$\sum_{n=0}^{\infty} \left(2c_n n + 2c_{n+1}(n+1) - c_n\right)x^n = 0.$$

$$c_{n+1} = \frac{c_n(1-2n)}{2(n+1)}$$

$$c_0 = c_0$$

$$c_1 = \frac{c_0}{2}$$

$$c_2 = \frac{-c_1}{2\cdot 2} = \frac{-c_0}{2^2\cdot 2!}$$

$$c_3 = \frac{-3c_2}{2\cdot 3} = \frac{3c_0}{2^3\cdot 3!}$$

$$c_4 = \frac{-5c_3}{2(4)} = \frac{-15c_0}{2^4\, 4!}$$

in general

$$c_n = \frac{c_0(2n-3)!!(-1)^{n+1}}{2^n n!}$$

for $n \geq 2$

The radius of convergence will be:

$$\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{c_0\,(2n-3)!!\,(-1)^{n+1}}{2^n\,n!}}{\frac{c_0\,(2n-1)!!\,(-1)^{n+2}}{2^{n+1}\,(n+1)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2\,(n+1)}{2n-1}\right| = 1.$$

So, the radius of convergence is 1.

$$y(x) = c_0\left(1+\frac{1}{2}x+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}c_0\,(2n-3)!!}{2^n\,n!}x^n\right)$$

Now,

$$\sqrt{1+x} = 1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)x^3}{3!}x^3+\cdots)$$

which we can rewrite as:

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}$$

$$= \frac{1\,(1-2)\,(1-4)\cdots(1-(2n-2))}{2^n\,n!}$$

$$= \frac{(-1)^{n+1}\,(2n-3)!!}{2^n\,n!}$$

So,
$$\boxed{y(x) = c_0\sqrt{1+x}}$$

8.1.13 $y'' + 9y = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y''(x) = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$$

So,

$$\sum_{n=0}^{\infty} c_n n(n-1) x^{n-2} + 9 \sum_{n=0}^{\infty} c_n x^n = 0$$

So, c_0 and c_1 can be chosen arbitrarily, and for the rest we get the relation:

$$\sum_{n=0}^{\infty} [c_{n+2}(n+2)(n+1) + 9c_n] x^n = 0$$

$$\Rightarrow \quad c_{n+2} = \frac{-9c_n}{(n+2)(n+1)}$$

This splits into odd and even terms:

Even:

$$c_0 = c_0$$

$$c_2 = \frac{-9c_0}{2 \cdot 1}$$

$$c_4 = \frac{-9c_1}{4 \cdot 3} = \frac{9^2 c_0}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$c_6 = \frac{-9^3 c_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \Rightarrow \cancel{c_n = (-1)^{n/2}(-1)} \quad c_{2n} = \frac{(-1)^n 3^{2n} c_0}{(2n)!}$$

Similarly,

$$c_{2n+1} = \frac{(-1)^n 3^{2n} x^{2n+1} c_1}{(2n+1)!}$$

Now, the radius of convergence for both of these will be:

$$\cancel{\lim_{n\to\infty} \left| \frac{c_{2n}}{c_{2n+2}} \right| = \lim \frac{}{n}}$$

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{k_n \, 3}{n} \right| = 0$$

where $k_n = \begin{cases} \frac{3c_0}{c_1} & \text{for } n \text{ even} \\ \frac{c_1}{3c_0} & \text{for } n \text{ odd} \end{cases}$

So, converges for all x and ROC is ∞.

The solutions are:

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!} \qquad y_2(x) = \frac{c_1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

which are:

$$y(x) = C_1 \cos(3x) + C_2 \sin(3x)$$

C_1, C_2 are unknown constants

8.1.21. $y'' - 2y' + y = 0; \quad y(0) = 0, \; y'(0) = 1$

$$y = \sum_{n=0}^{\infty} c_n x^n$$
$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$$
$$y'' = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2} - 2\sum_{n=0}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

c_0 and c_1 are "arbitrary" in that they're just determined by the initial conditions.

The other coefficients are determined by the recursion relation:

$$c_{n+2}(n+2)(n+1) - 2c_{n+1}(n+1) + c_n = 0$$

$$\Rightarrow \boxed{c_{n+2} = \frac{2c_{n+1}(n+1) - c_n}{(n+2)(n+1)}}$$

Now, $y(0) = c_0 = 0.$

$y'(0) = c_1 = \underline{1}.$

The higher order terms are given by:

$$c_2 = \frac{2}{2} = 1 = \frac{1}{1!}$$

$$c_3 = \frac{2 \cdot 1 \cdot 2 - 1}{3 \cdot 2} = \frac{1}{2} = \frac{1}{2!}$$

$$c_4 = \frac{2 \cdot \left(\frac{1}{2}\right) \cdot 3 - 1}{4 \cdot 3} = \frac{1}{6} = \frac{1}{3!}$$

$$c_5 = \frac{2 \cdot \left(\frac{1}{6}\right) \cdot 4 - \frac{1}{2}}{5 \cdot 4} = \frac{5/6}{5 \cdot 4} = \frac{1}{24} = \frac{1}{4!}$$

$$c_6 = \frac{2 \cdot \left(\frac{1}{24}\right) \cdot 5 - \frac{1}{6}}{6 \cdot 5} = \frac{1}{6 \cdot 5 \cdot 4} = \frac{1}{5!}$$

and in general:

$$c_n = \frac{1}{(n-1)!}$$

This can be proven by noting
Base case:

$$c_2 = \frac{1}{(2-1)!} = 1 \quad \text{which is true.}$$

~~Induction:~~

$$\cancel{2\left(\frac{1}{n!}\right)(}$$

Induction:

Assume it's true up to $n+1$:

$$c_{n+2} = \frac{2\,c_{n+1}\,(n+1) - c_n}{(n+2)(n+1)}$$

$$= \frac{2\left(\frac{1}{n!}\right)(n+1) - \frac{1}{(n-1)!}}{(n+2)(n+1)}$$

$$= \frac{\frac{2}{(n-1)!} + \frac{2}{n!} - \frac{1}{(n-1)!}}{(n+2)(n+1)}$$

$$= \frac{\frac{2n+2-n}{n!}}{(n+2)(n+1)} = \frac{(n+2)}{(n+2)(n+1)\,n!}$$

$$= \frac{1}{(n+1)!}$$

So, it's true for c_{n+2}.

Therefore,

$$c_n = \frac{1}{(n-1)!}$$

So, we get:

$$y(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

$$= x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$= x \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \boxed{x e^x}$$

8.1.25. For the initial value problem:

$$y'' = y' + y \quad ; \quad y(0) = 0, \; y(1) = 1$$

derive the power series solution

$$y(x) = \sum_{n=1}^{\infty} \frac{F_n}{n!} x$$

where $\{F_n\}_{n=0}^{\infty}$ is the sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-2} + F_{n-1}$ for $n > 1$.

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} c_n n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$$

Plugging this into our ODE we get:

$$\sum_{n=0}^{\infty} c_n\, n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n\, n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Now c_0 and c_1 are determined by their initial conditions.

$$y(0) = c_0 = 0.$$

$$y(1) = \sum_{n=1}^{\infty} c_n = 1.$$

Now, for $n \geq 2$ we have the recursion relation:

$$\sum_{n=0}^{\infty} [c_{n+2}(n+2)(n+1) - c_{n+1}(n+1) - c_n]x^n = 0.$$

So, $$c_{n+2} = \frac{c_{n+1}(n+1) + c_n}{(n+2)(n+1)}$$

Now, if we assume it's true for up to $n+1$,

$$c_{n+2} = \frac{\frac{F_{n+1}}{(n+1)!}(n+1) + \frac{F_n}{n!}}{(n+2)(n+1)} = \frac{F_{n+1} + F_n}{(n+2)!} = \frac{F_{n+2}}{(n+2)!}$$

So, it's true inductively.

We still need to prove that $c_1 = 1$.

So, we point out that:

$$y(1) = \sum_{n=1}^{\infty} c_n \quad \text{if } c_1 = 1$$

then if c_k for $k > 1$ are not all 0 then there is <u>no way</u> $y(1) = 1$.

This is <u>another</u> typo in the textbook.

It <u>should</u> be $y'(0) = 1$.

In this case $y'(0) = c_1 = 1$.