REBEL ALGEBRAIC GEOMETRY SEMINAR
MORE PROJECTIVE VARIETIES

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1. Introduction

Last time we introduced *projective varieties*, which are essentially a blending of our notion of "affine variety" together with our recognition of the fact that $k[x_1, ..., x_n]$ is a graded ring. Today we're going to look more at the relationships between projective and affine varieties.

2. Projective space is locally affine

Let k be a field. Recall that projective n-space, denoted $\mathbb{P}^n$, is the space

$$k^{n+1} \setminus \{0\}/\sim \text{ where } \nu \sim \lambda\nu \text{ for all } \lambda \in k^\times.$$

We have given $\mathbb{P}^n$ the Zariski topology by defining our closed sets to be the zero sets of collections of homogenous polynomials in $k[x_0, ..., x_n]$. We're now going to show that, topologically, $\mathbb{P}^n$ is locally just $\mathbb{A}^n$.

For $i = 0, ..., n$, let $U_i \subseteq \mathbb{P}^n$ be the set of point

$$U_i = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n : a_i \neq 0\}.$$

Define a map $\varphi_i : U_i \to \mathbb{A}^n$ by

$$U_i(a_0 : \cdots : a_n) = \left(\frac{a_0}{a_i}, \cdots, \frac{a_n}{a_i}\right),$$

where the $\frac{a_i}{a_i}$ term is omitted.

We first observe that the set $\{U_i\}$ covers $\mathbb{P}^n$. To see this, note that any point in $\mathbb{P}^n$ must have a non-zero coordinate. If this is the i-th coordinate, then this point is in U_i. We also note that each U_i is open in the Zariski topology, since U_i is the complement of the zero set of the polynomial x_i.

Next we note that the φ_i are bijections. To see this, define $\psi_i : \mathbb{A}^n \to U_i$ by

$$\psi_i(a_0, ..., a_n) = (a_0 : \cdots : 1 : \cdots a_n)$$

where the i-th position on the left hand side is omitted, and the i-th position on the right hand side is just 1. One can quickly see that ψ_i and φ_i are inverses of one another.

In fact, φ_i and ψ_i are continuous, and thus φ_i is a homeomorphism. To see this, it is enough to check that φ_i and ψ_i are both closed maps (that is,

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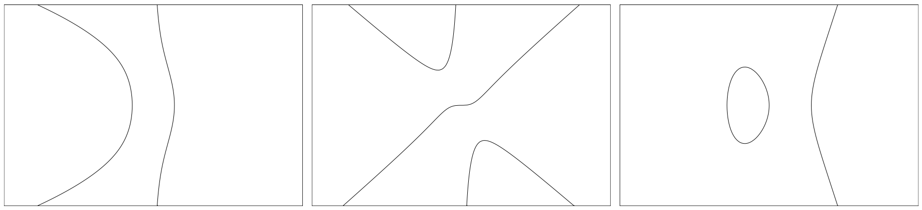

FIGURE 1. E_x, E_y, and E_z, respectively. The figure for E_z is perhaps the most familiar image of an elliptic curve, though it only represents the curve in a single affine patch.

they map closed sets to closed sets). We'll check this for $i = 0$, noting that the general case is the same after relabeling some indices.

If $Y \subseteq \mathbb{A}^n$ is a closed set, then it is given as the zero set of an ideal of polynomials $\mathfrak{a} \subseteq k[x_1, ..., x_n]$. If we let $\mathfrak{b} \subseteq k[x_0, ..., x_n]$ be the homogenization of $\mathfrak{a}$ by introducing x_0, then $\psi_0(Y) = Z(\mathfrak{b})$, which is closed.

Similarly, if $X \subseteq U_0$ is closed, then X is given as the zero set of a collection T of homogenous polynomials in $k[x_0, ..., x_n]$. We can send T to $\tilde{T} \subseteq k[x_1, ..., x_n]$ simply by sending $x_0 \mapsto 1$. One now checks that $\varphi_0(X) = Z(\tilde{T})$, which is closed.

3. Example: An elliptic curve

We've just shown that locally the topology on $\mathbb{P}^n$ is just $\mathbb{A}^n$. Let's look at an example of this.

Consider the polynomial $E = -zy^2 + x^3 - 2xz^2$. This polynomial is homogeneous and has 3 variables, and thus it should correspond to a closed subset of $\mathbb{P}^2$, which by abuse of notation we'll also denote by E. We'll take $k = \mathbb{R}$ so that we can draw pictures.

Projective space $\mathbb{P}^2_{\mathbb{R}}$ is unwieldy to draw, but by the work done above, we know that we can draw $\mathbb{P}^2_{\mathbb{R}}$ in pieces. The three pieces are U_x, U_y, U_z. Let's push E through the homeomorphisms $\varphi_x, \varphi_y, \varphi_z$, respectively.

Recall that each of these homeomorphisms acts on E by essentially substituting 1 for one of the variables. We thus obtain three polynomials, each corresponding to a different affine patch of $\mathbb{P}^2$:

$$\begin{aligned} E_x &: -zy^2 + 1 - 2z^2, \\ E_y &: -z + x^3 - 2xz^2, \\ E_z &: -y^2 + x^3 - 2x. \end{aligned}$$

These three patches are shown side-by-side in Figure 1.