# REBEL ALGEBRAIC GEOMETRY SEMINAR
# PROJECTIVE VARIETIES

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## 1. Introduction

We've already talked about *affine varieties*, which are essentially constructed by taking the quotient of a polynomial ring by an ideal. Of course, ideals come in many flavors, sometimes carrying a great deal of special information. It is therefore sometimes (often!) helpful to consider alternative constructions for use in these special cases. Naturally, we will want there to be some compatibility between the original construction and these specialized constructions, and indeed for what we are doing today this will be the case.

## 2. Graded rings

We first need to introduce some additional algebraic language. Let $S$ be a ring. We say that $S$ is a *graded ring* if $(S, +)$ has a decomposition (as an abelian group) of the form

$$S = \bigoplus_{d \geq 0} S_d,$$

where for any $d, e \geq 0$ we have $S_d \cdot S_e \subseteq S_{d+e}$. Put another way, $S$ is graded if we can split up $S$ into different "tiers" where, if $a, b \in S$ are both in $S_d$, then $a + b$ is also in $S_d$, and if $a \in S_d$ and $b \in S_e$, then $ab \in S_{d+e}$.

Graded rings arise quite often. One good sign that you're working with a graded ring is if the elements of your ring have a notion of "degree" that behaves like the degree of a polynomial. Indeed, this will be the situation we are primarily concerned with.

Let $S = k[x_0, ..., x_n]$. We'll show that this is a graded ring. For $d \geq 0$, let

$$S_d = k\Big\{x_0^{i_0} \cdots x_n^{i_n} : i_0 + \cdots + i_n = d\Big\},$$

where we interpret the right hand side to mean "all $k$-linear sums of monomials whose total degree is $d$." For instance, $4x_0^2x_3^5 + 2x_0x_1x_2x_3^4 \in S_7$. For any fixed $d$, $S_d$ is an abelian group under addition, and certainly $S = \bigoplus_{d \geq 0} S_d$. Finally, if $f(x) \in S_d$ and $g(x) \in S_e$, then every term in $f(x)$ has total degree $d$, and every term in $g(x)$ has total degree $e$, so that every term of $f(x)g(x)$ has total degree $d + e$. Thus $S$ is a graded ring.

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We'll now pause to introduce a useful bit of notation. Let

$$x_0^{\alpha_0} \cdots x_n^{\alpha_n}$$

be a monomial in $S$. Let $\alpha = (\alpha_0, ..., \alpha_n)$. Notationally, we let

$$\mathbf{x}^\alpha = x_0^{i_0} \cdots x_n^{i_n}.$$

If one defines the convention that $\alpha + \beta = (\alpha_0 + \beta_0, ..., \alpha_n + \beta_n)$, then we obtain the pleasing identity

$$\mathbf{x}^\alpha \cdot \mathbf{x}^\beta = \mathbf{x}^{\alpha+\beta}.$$

One often writes $\|\alpha\| = \alpha_0 + \cdots + \alpha_n$, so that we can express the fact that $S$ is graded by degree in the following way: if $\mathbf{x}^\alpha \in S_d$ and $\mathbf{x}^\beta \in S_e$, then $\|\alpha\| = d$ and $\|\beta\| = e$, so that $\|\alpha + \beta\| = d + e$, and thus $\mathbf{x}^\alpha \cdot \mathbf{x}^\beta = \mathbf{x}^{\alpha+\beta} \in S_{d+e}$.

If $S$ is graded, we say an element $f \in S$ is *homogeneous* if $f \in S_d$ for some $d \geq 0$. Writing $f = \sum_{i=1}^N f_i \mathbf{x}^{\alpha_i}$, $f$ is homogeneous if and only if $\|\alpha_i\| = \|\alpha_j\|$ for all $i, j$. For instance, $x_0^2 x_1 - x_0 x_1 x_2$ is homogenous, whereas $x_0^2 x_1 - x_0 x_1 x_2^2$ is not.

When a ring $S$ is graded, there is a special class of ideals which have some added structure that is compatible with the grading. We say an ideal $\mathfrak{a} \subseteq S$ is *homogeneous* if $\mathfrak{a}$ can be generated by homogenous elements of $S$. Note that it may also be possible that $\mathfrak{a}$ can be generated by elements which are *not* homogeneous, but this is irrelevant. Equivalently, $\mathfrak{a}$ is homogeneous if and only if

$$\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d)$$

as abelian groups.

**Exercise 2.1.** *Prove this last claim.*

## 3. Projective space

Let $k$ be a field. We define *projective $n$-space* over $k$, denoted $\mathbb{P}_k^n$ (or just $\mathbb{P}^n$ if there will be no confusion) to be the quotient space

$$(k^{n+1} \setminus \{0\}) / \sim \text{ where } \nu \sim \lambda\nu \text{ for all } \lambda \in k^\times.$$

Recall that polynomials in $k[x_1, ..., x_n]$ define functions $\mathbb{A}_k^n \to k$. Unfortunately, polynomials in $S = k[x_0, ..., x_n]$ do *not* define functions $\mathbb{P}_k^n \to k$. As an example of this, let $f(x_0, ..., x_n) = x_0$. Since $(1 : 0 : \cdots : 0) = (2 : 0 : \cdots : 0)$, we'd like to say that $f(1 : 0 : \cdots : 0) = f(2 : 0 : \cdots : 0)$, but this clearly is not the case. Luckily, we're only interested in the zero sets of polynomials, and as it happens, this *is* a well-defined notion for certain types of polynomials.

A polynomial $f(\mathbf{x}) \in S$ is said to be *homogeneous of degree $d$* if $f \in S_d$. In this case, note that

$$f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x}),$$

a fact we can see by checking that this property holds for monomials in $S_d$. In particular, if $f(\mathbf{x}) = 0$, then $f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x}) = 0$, so that the following definition is well-defined: if $T \subseteq S$ is a collection of homogeneous polynomials, we let

$$Z(T) = \{p \in \mathbb{P}^n : f(p) = 0 \text{ for all } f \in T\}.$$

If $\mathfrak{a}$ is a homogeneous ideal in $S$, we define $Z(\mathfrak{a}) = Z(T)$, where $T$ is the set of homogeneous elements of $\mathfrak{a}$ (note that, in general, $T \subsetneq \mathfrak{a}$.)

A subset $Y \subseteq \mathbb{P}^n$ is said to be an *algebraic set* if there exists a collection $T \subseteq S$ of homogeneous polynomials such that $Y = Z(T)$.

**Exercise 3.1.** *Show that*

*(a) The empty set and $\mathbb{P}^n$ are both algebraic sets.*
*(b) The union of two algebraic sets is an algebraic set.*
*(c) The intersection of any family of algebraic sets is again an algebraic set.*

Once more, we define the *Zariski topology* on $\mathbb{P}^n$ by taking the open sets to be the compliments of algebraic sets. With this topology, a *projective algebraic variety*, or more simply a *projective variety*, is an irreducible algebraic subset of $\mathbb{P}^n$ with the subspace topology.

If $Y \subseteq \mathbb{P}^n$ is any subset, we define the *homogeneous ideal of* $Y$, denoted $\mathbb{I}(Y)$, to be the ideal generated by

$$\{f \in S : f \text{ is homogeneous and } f(p) = 0 \text{ for all } p \in Y\}.$$

## 4. Compatibility with affine varieties

If $S$ is a graded ring, we've got two spaces associated to it: both $\mathbb{P}^n$, built with the use of the graded structure of $S$, and $\mathbb{A}^{n+1}$, built without any knowledge of this structure. Naturally, we'd like to know what sorts of relationships there are between $\mathbb{P}^n$ and $\mathbb{A}^{n+1}$. In practice, it is actually easier to compare $\mathbb{P}^n$ with $\mathbb{A}^n$, which is what we'll now set out to do.

For $i = 0, ..., n$, let $U_i \subseteq \mathbb{P}^n$ be the set of points

$$U_i = \{(a_0 : ... : a_n) \in \mathbb{P}^n : a_i \neq 0\}.$$

Define a map $\varphi_i : U_i \to \mathbb{A}^n$ by

$$U_i(a_0 : ... : a_n) = \left(\frac{a_0}{a_i}, ..., \frac{a_n}{a_i}\right),$$

where the term $\frac{a_i}{a_i}$ is omitted.

**Exercise 4.1.** *Show that*

*(a) The set $\{U_i\}_{i=0}^n$ covers $\mathbb{P}^n$, and*
*(b) Giving $U_i$ the subspace topology, $\varphi_i : U_i \to \mathbb{A}^n$ is a homeomorphism.*