UNCTIONS on THE AFFINE SPACE

et $f \in R$, we define $\varphi = \varphi_f$ as $\begin{array}{l} \mathbb{A}^n \xrightarrow{\varphi} K = \mathbb{A}^1 \\ p \longmapsto f(p) \end{array}$

ERCISE: PROVE THA φ is continuous

f: A map $\psi: \mathbb{A}^n \longrightarrow \mathbb{A}^1$ is a regular function if $\exists f \in R$ such that $p \in \mathbb{A}^n$, $\psi(p) = f(p)$ $[\Leftrightarrow \psi = \varphi_f, \exists f]$

t us now consider $X \subseteq \mathbb{A}^n$ closed, we want to define regular nctions on X.

et $f \in R$, and $\varphi_f: X \longrightarrow \mathbb{A}^1$ continuous.
$$p \longmapsto f(p)$$

: $g \in \mathbb{I}(X)$ $\quad \varphi_{f+g}: X \longrightarrow \mathbb{A}^1$
$$p \longmapsto (f+g)(p) = f(p) + g(p) = f(p) \quad \Big| \Rightarrow \varphi_f = \varphi_{f+g}$$

nere is no 1:1 correspondence between regular functions and polynomials.

f: Let $\psi: X \longrightarrow \mathbb{A}^1$; ψ is a regular function if $\underline{\exists}$ $f \in R$ s.t. $\forall p \in X$, $\psi(p) = f(p)$

e denote $\mathbb{K}[X] := \{\varphi: X \longrightarrow \mathbb{A}^1 \mid \varphi \text{ is regular}\}$ the set of ALL egular functions on X.

S: ① $\mathbb{K}[p] = K$
$\forall$ closed X, $K \subseteq \mathbb{K}[X]$

e have a map $\quad \mathbb{K}[X_1,\dots X_n] \longrightarrow \mathbb{K}[X]$

$$f \longmapsto \varphi_f$$

$-g \in \mathbb{I}(X) \iff \varphi_f = \varphi_g$ in fact $f-g \in \mathbb{I}(X) \iff \forall p \in X \quad f(p)=g(p)$

$$\iff \varphi_f(p)=\varphi_g(p) \iff \varphi_f=\varphi_g$$

OBTAIN:

$$\begin{tikzcd} R=[X_1,\dots X_n] \arrow[r] \arrow[d] & \mathbb{K}[X] \\ R/\mathbb{I}(X) \arrow[ur, "\cong", "F"'] & \end{tikzcd}$$

where F is a NATURAL ISOMORPHISM of RINGS ($\mathbb{K}$-algebras)

: ① $\mathbb{K}[X]$ has no nilpotents

.e. $\forall \varphi \in \mathbb{K}[X]$, $\forall n \in \mathbb{N}$, $\varphi^n \neq 0$ as $\varphi \neq 0$ (because $\mathbb{I}$ is radical!)

) $\mathbb{K}[X]$ is a domain $\iff$ X is IRREDUCIBLE ($\mathbb{I}$ prime)

f: Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ closed sets. A regular map from X to Y is

. map $\quad \Phi: X \longrightarrow Y$

$$p \longmapsto (\varphi_1(p),\dots \varphi_m(p)) \qquad \text{such that} \quad \varphi_i \in \mathbb{K}[X]$$

f: An ISOMORPHISM between X and Y is a regular map $\Phi: X \longrightarrow Y$

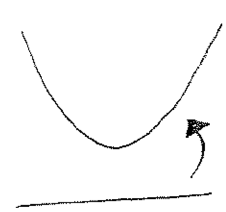
at admits an inverse $\Phi^{-1}: Y \longrightarrow X$, where Φ^{-1} is regular and

$\Phi \circ \Phi^{-1} = Id_Y \qquad \Phi^{-1} \circ \Phi = Id_X$

XAMPLES:

) Let $Z \subseteq \mathbb{A}^2$, $Z = Z(y-x^2)$ $\qquad \Phi: \mathbb{A}^1 \longrightarrow Z \subseteq \mathbb{A}^2$

$$t \longmapsto (t,t^2)$$

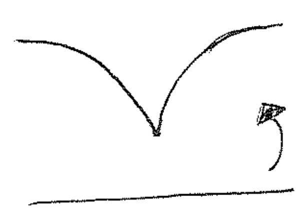


has a regular inverse, the projection on the

irst cohordinate $\quad \mathbb{A}^2 \longrightarrow \mathbb{A}^1$, $(x,y) \longmapsto x \qquad [(t,t^2) \longmapsto t]$

, $\Phi: \mathbb{A}^1 \longrightarrow Z \subseteq \mathbb{A}^2$ $\qquad$ regular and 1 to 1 onto Z

$$t \longmapsto (t^3,t^2)$$



IOWEVER it is NOT an ISOMORPHISM!

don't remember if we already did it:

: For any finitely generated K-algebra A, $\exists n$ and $I \subseteq k[x_1, \dots x_n]$ eal such that $A \cong k[x_1, \dots x_n]/I$

t $X \subseteq \mathbb{A}^n$ closed, $\mathbb{I}(X) = I$ $X = Z(I)$

$[X]$ is a finitely generated K-algebra over k and without lpotents, in fact $\mathbb{K}[X] = k[x_1, \dots x_n]/I$

EMMA: ① Let $\Phi : X \longrightarrow Y$ regular map. Then there exists a ique homomorphism of K-algebras:

$$\Phi^* : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$$
$$\psi \longmapsto \psi \circ \Phi$$

(viceversa) Given $H : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$, $\exists!$ regular map $\Phi : X \longrightarrow Y$ such that $H = \Phi^*$

A regular map $\Phi : X \longrightarrow Y$ is an isomorphism iff $\Phi^* : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$ is an isomorphism.

: ① $\Phi : X \longrightarrow Y$ map between closed

$p \longmapsto (\varphi_1(p), \dots \varphi_m(p))$ $\varphi_i \in \mathbb{K}[X] = k[x_1, \dots x_n]/\mathbb{I}(X)$

$\Phi^* : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$

$\psi \longrightarrow \psi \circ \Phi : X \longrightarrow Y \longrightarrow K$ $\psi \in \mathbb{K}[Y] = k[y_1, \dots y_m]/\mathbb{I}(Y)$

et φ_i be represented by the poly $f_i(x_1, \dots x_n) \in k[x_1, \dots x_n]$

et ψ be represented by $g(y_1, \dots y_m)$

given $p \in X$:

$\psi(\Phi(p)) = g(\varphi_1(p), \dots \varphi_m(p)) = g(f_1(p), \dots f_m(p))$ and using $p = (x_1, \dots x_n)$

t is clear that $g(f_1(p), \dots f_m(p)) \in k[x_1, \dots x_n]$

et $\bar{y}_i$ generate $\mathbb{K}[Y]$ over k, then $\bar{y}_i \longrightarrow \bar{f}_i(x_1, \dots x_n) \in \mathbb{K}[X]$

OK!

EXAMPLE: $X = \mathbb{A}^1 \xrightarrow{\Phi} Y = Z(x^2 - y^3) \subseteq \mathbb{A}^2$

$t \longmapsto (t^3, t^2)$

$\quad\quad\quad\quad f_1 \quad f_2$

$$\mathbb{K}[Y] = \mathbb{K}[x,y]/(x^2-y^3) \xrightarrow{\Phi^*} \mathbb{K}[\mathbb{A}^1] = \mathbb{K}[t]$$

$$\bar{x} \longmapsto t^3$$

$$\bar{y} \longmapsto t^2$$

Im $\Phi^* \subseteq \mathbb{K}[t]$ but $t \notin \mathrm{Im}(\Phi^*) \rightsquigarrow$ not an isomorphism!

$\Rightarrow \Phi$ is not an isomorphism!

②) Given $H: \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$

$\bar{y}_i \longmapsto \bar{f}_i \quad i = 1, \dots m$

Let us first consider $\Phi: X \longrightarrow \mathbb{A}^m$

$p \longmapsto (f_1(p), \dots f_m(p))$ for any f_i representing $\bar{f}_i$

Is $\Phi(X) \subseteq Y$? $\forall g \in \mathbb{I}(Y)$, $g|_{\Phi(X)} \equiv 0$, that is $\forall p \in X \quad g(\Phi(p)) = 0$

but this is trivial because $f_i = \bar{f}_i + t$ w/ $t \in I(X)$ and $\mathrm{Im}(g) = \bar{0} + t$

③ $\Rightarrow$ If Φ is an isomorphism $\Phi^{-1}: Y \longrightarrow X$ is regular,

$\Phi \circ \Phi^{-1} = Id_Y$, $\Phi^{-1} \circ \Phi = Id_X$

Then $(\Phi^{-1})^*: \mathbb{K}[X] \longrightarrow \mathbb{K}[Y]$ and by definition

$\Phi^*(\Phi^{-1})^* = Id_{\mathbb{K}[X]}$

$\Leftarrow$ Follows from ② in the same way.

OBS: We have an equivalence of categories:

$$\left\{\begin{array}{l}\text{Finitely generated algebras} \\ \text{over } \mathbb{K}, \text{ without nilpotents}\end{array}\right\} \Big/ \text{Iso} \longleftrightarrow \left\{\begin{array}{l}\text{Affine closed} \\ \text{over } \mathbb{K}\end{array}\right\} \Big/ \text{Iso}$$

AMPLE: Let $\Phi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ w/ $p = \mathrm{char}(K)$

$(a_1, \dots a_n) \mapsto (a_1^p, \dots a_n^p)$

ROBENIUS MORPHISM. Φ is one to one and surjective.

$a^p - b^p = 0 \iff (a-b)^p = 0 \iff a = b$)

IT AN ISOMORPHISM?

$\Phi^* : K[x_1, \dots x_n] \longrightarrow K[x_1, \dots x_n] \Rightarrow \mathrm{Im}\,\Phi^* = K[x_1^p, \dots x_n^p] \subsetneq K[x_1, \dots x_n]$

$x_i \longmapsto x_i^p$

NOT AN ISOMORPHISM.

ODUCTS:

e want to consider $\mathbb{A}^n \times \mathbb{A}^m$

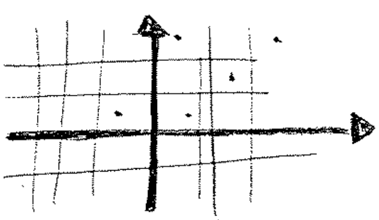
e have that, has sets $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$; in fact $K^n \times K^m = K^{n+m}$.

owever the product topology over K^{n+m} is <u>NOT</u> the Zariski

pology of $\mathbb{A}^{n+m}$!!

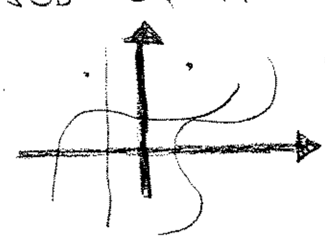
<u>x</u>: $n = m = 1$

CLOSED of $\mathbb{A}^1 \times \mathbb{A}^1$:



only points and straight lines

• CLOSED of $\mathbb{A}^2$



also all the other curves

PRODUCT TOPOLOGY $\subsetneq$ ZARISKI TOPOLOGY

$\boxed{\text{e will say that } \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m} \text{ w/ the Zariski topology!}}$

IRST EXAMPLE: Let $\mathbb{A}^n_{\underline{x}} \times \mathbb{A}^n_{\underline{y}} = \mathbb{A}^{2n} = K[x_1, \dots x_n, y_1, \dots y_n]$

et us consider $\Delta \subseteq \mathbb{A}^n \times \mathbb{A}^n$ $\quad \Delta = Z(x_i - y_i,\ i = 1, \dots n)$ THE DIAGONAL

$K[\Delta] = K[\underline{x}, \underline{y}]/(x_i - y_i,\ i = 1, \dots n) \cong K[x_1, \dots x_n] = K[\mathbb{A}^n]$

ote: Δ is not in the PRODUCT TOPOLOGY.

* $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$

$X \times Y$ is an affine closed of $\mathbb{A}^{n+m}$ and

$X \times Y = \mathcal{Z}(\mathbb{I}(X), \mathbb{I}(Y))$

$\mathbb{K}[X \times Y] \cong \mathbb{K}[X] \otimes_K \mathbb{K}[Y]$

A regular function on $X \times Y$ is in the form

$$h(\underline{x}, \underline{y}) = \sum \lambda_i f_i(\underline{x}) g_i(\underline{y}), \quad \lambda_i \in k, \quad f_i \in \mathbb{K}[X], \quad g_i \in \mathbb{K}[Y].$$

<u>ERCISE:</u> Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, $\varphi: X \longrightarrow Y \subseteq \mathbb{A}^m$ regular.

e graph of φ is

$(X \times \mathbb{A}^m \supseteq) \Gamma_\varphi := \{(p,q) \mid p \in X, q \in \mathbb{A}^m \text{ s.t. } q = \varphi(p)\} = \{(p, \varphi(p)) \mid p \in X\}$

OVE THAT Γ is a closed in $X \times \mathbb{A}^m$ and it is ISOMORPHIC to X

<u>E</u>: Let us denote $\mathbb{A}^n_f = \mathbb{A}^n \setminus \mathcal{Z}(f)$

ore in general, given $X \subseteq \mathbb{A}^n$, $X_f = \{p \in X \mid f(p) \neq 0\}$

<u>ERCISE</u> (NOT EASY) $\forall f$ PROVE THAT $\mathbb{A}^n_f$ is ISOMORPHIC to

e AFFINE CLOSED $\mathcal{Z} = \mathcal{Z}(1 - f \cdot y) \subseteq \mathbb{A}^{n+1}$ $[K[x_1, \dots x_n, y]]$

<u>f</u>: • The open in the form $\mathbb{A}^n_f$ (or X_f) are called PRINCIPAL

e closed of $\mathbb{A}^n$ corresponding to principal ideals are called

ypersurfaces $[\mathbb{I}(X) = (f)]$

<u>S</u>: The principal open are a base for the Zariski topology.

<u>MMA</u>: The projection $\mathbb{A}^{n+m} \longrightarrow \mathbb{A}^m$ is OPEN

$(p,q) \longmapsto p$

<u>OROLLARY</u>: the projection $X \times Y \longrightarrow Y$ is OPEN (for

X, Y closed)

ef: A regular map between closed $\varphi : X \to Y$ is said to
dominant is $\varphi(X)$ is DENSE in Y $\left(\overline{\varphi(X)} = Y\right)$

oPOSITION: Let X, Y irreducible. φ is dominant $\iff$ φ^* is INJECTIVE

) Let φ^* injective
n absurd, suppose $\exists Z$ closed, $Z \subsetneq Y : \varphi(X) \subseteq Z$
et $f \in \mathbb{K}[Y]$, $f \neq 0$ s.t. $f|_Z = 0$

en: $\varphi^*(f) = f \circ \varphi$ and $\forall p \in X$ $\varphi^*(f)(p) = f \circ \varphi(p) = 0$
$\Rightarrow \varphi^*(f) = 0$
because of the injectivity $\divideontimes$

) Let $\overline{\varphi(X)} = Y$
n absurd, suppose $\exists f \in K[Y]$ $f \neq 0$ s.t. $\varphi^*(f) = 0$

$^*(f) = f \circ \varphi$. $\forall p \in X$ $\varphi^*(f)(p) = f(\varphi(p)) = 0$
$\Rightarrow \varphi(X) \subset Z(f) \underset{\substack{\downarrow \\ f \neq 0}}{\subsetneq} Y$ $\divideontimes$

ERCISES:

PROVE THAT $\phi : \mathbb{A}^1 \to \mathbb{A}^n$, $t \mapsto (t, t^2, \ldots t^n)$ is an isomorphism onto its image.

PROVE THAT $X = Z(xy - 1)$ and $Y = Z(y - x^2) \subseteq \mathbb{A}^2$ are not ISOMORPHIC

Let X be the set given by two points. Consider it as an ffine closed. Describe $\mathbb{K}[X]$ and repeat the xercise for a finite set of points.