AFFINE ALGEBRAIC GEOMETRY

Let k be a field. Let $R = k[x_1, \dots x_n]$ $n \geq 0$

The ambient space is $k^n = \{(a_1, \dots a_n), a_i \in k\}$ as a set.

In this space we want to introduce a topology (a geometry)

$Z(f) := \{p \in k^n : f(p) = 0\}$ for $f \in R$

MORE in general $Z(T) := \{p \in k^n : f(p) = 0\}$ for $T \subseteq R$

① $n = 1$ $k = \mathbb{R}$ $f \in \mathbb{R}[x]$ $f \neq c$ $Z(f)$ is a finite set of points in $\mathbb{R}$

we prefer to have an algebraically closed field so that

$Z(f) \neq \phi$ $\forall f$ non-constant

NOTE: Let $k = \bar{k}$ $\forall f \in R \setminus k$ $Z(f)$ is nonempty & if $n \geq 2$ $Z(f)$ is infinite

PF: suppose $n = 2$ $f(x, y) \notin k$.

$\forall a \in k$ $f(a, y)$ is a poly of one variable $\Rightarrow$ admits zeros in the form (a, b_i) $i \in I$.

If $f(a, y) \in k$ $f = (x - a)\tilde{f}(x, y) + c$ $f(a, y) = c \in k$

but this can be true only for finite values. OK!

DEF / PRO: The "ZARISKI TOPOLOGY" over k^n has as set of closed the class $\mathcal{C} = \{Z(T) \quad \forall T \subseteq k[x_1, \dots x_n]\}$

PF: ① $\phi, k^n \in \mathcal{C}$ $\Leftarrow$ $\phi = Z(1) = Z(\lambda)$ $\forall \lambda \in k^*$; $k^n = Z(0)$

② $\mathcal{C}$ is closed respect to finite union

$\Rightarrow Z(T_1) \cup Z(T_2) = Z(T_1 \cdot T_2)$ where $T_1 \cdot T_2 = \{f \cdot g \mid f \in T_1, g \in T_2\}$

we have $Z(T_1 \cdot T_2) \subset Z(T_1) \cup Z(T_2)$

for every point p, $(fg)(p) = 0 \Rightarrow f(p) \cdot g(p) = 0$

If $p \notin Z(T_1) \Rightarrow f(p) \neq 0 \Rightarrow g(p) = 0 \Rightarrow p \in Z(T_2) \Rightarrow p \in Z(T_1) \cup Z(T_2)$

$\supseteq$ is obvious.

③ $\mathcal{C}$ is closed respect arbitrary intersection: SIMILARLY:

$\bigcap_{i \in I} Z(T_i) = Z(\bigcup_{i \in I} T_i)$ $Z(X) \cap Z(Y) = Z(X, Y)$ EASY OK!

DEF: k^n w/ ZARISKI TOPOLOGY is said AFFINE SPACE OF DIMENSION n OVER k and denoted $\mathbb{A}^n_k$ (or $\mathbb{A}^n$)

PROPERTY: The points are closed: $p = (a_1, \dots a_n) = Z(x_1 - a_1, \dots, x_n - a_n)$

EX: If $T_1, T_2 \subset R$ and $T_1 \subset T_2 \Rightarrow Z(T_1) \supset Z(T_2)$ -
- MAKE AN EXAMPLE FOR WHICH THE OPPOSITE is FALSE -

Let k to be infinite, let $U \subset \mathbb{A}^1$, $U \neq \emptyset$ open $\Rightarrow$
② $\bar{U} = \mathbb{A}^1$ ① $\forall$ open $V \subseteq \mathbb{A}^1$, $V \neq \emptyset \Rightarrow U \cap V \neq \emptyset$

Pf (①): If $U \cap V = \emptyset \Rightarrow U \subseteq V^c$ that is finite because it is closed
$\Rightarrow U$ is finite $\divideontimes$ (k infinite & $k = U \cup U^c$)

③ If $U \subseteq \mathbb{A}^n$ open, $U \neq \emptyset \Rightarrow \bar{U} = \mathbb{A}^n$

LEMMA: Let $f, g \in k[x,y]$ irreducible and non-null w/ $f \neq \lambda g$ $\lambda \in k$ then $Z(f,g)$ is a finite set of points.

Pf: Suppose that y appears in f ($f \notin k[x]$), let us consider

$$\underset{\substack{\shortparallel \\ k[x][y] \\ \text{domain}}}{k[x,y]} \subseteq \underset{\substack{\shortparallel \\ F[y]}}{k(x)[y]} \text{ euclidean ring}$$

where F is the field of fractions of $k[x]$

f and g are still irreducible in $k(x)[y]$ and
$1 = \alpha(x,y) f(x,y) + \beta(x,y)\, g(x,y)$ w/ $\alpha, \beta \in k(x)[y]$
CANCELLING THE DENOMINATOR WE OBTAIN $d(x) = a(x,y) f + b(x,y)\, g$.
If $(x_0, y_0) \in Z(f,g) \Rightarrow x_0 \in Z(d)$. Because $Z(d)$ is finite, x_0 varies in a finite set. If we fix $x_0 \Rightarrow f(x_0, y) \in k[y]$ & $(x - x_0) \nmid f$ because f is irredux $\Rightarrow f(x_0, y) \neq 0$. Has finite zeros in y.
$\Rightarrow Z(f(x_0, y))$ is finite $\Rightarrow Z(f(x_0,y), g(x_0,y))$ is finite ok!

COR: If $Z \in \mathbb{A}^2$, Z closed $\Rightarrow Z = Z(f) \cup \{P_1, \dots P_n\}$ $n \geq 0$

LET'S GO BACK TO THE NULLSTELLENSATZ

T generates an ideal $(T) = I = \{\sum f_i g_i \quad f_i \in T, g_i \in R\}$

OBSERVATION: $Z(T) = Z(I)$

$\subset$ obvious, $\supset$ if $T \subset I \Rightarrow Z(T) \supset Z(I)$

As we said we have a 1:1 correspondence

$$\{\text{Radical ideals of } R\} \underset{\mathbb{I}}{\overset{Z}{\rightleftarrows}} \{\text{closed of } \mathbb{A}^n\}$$

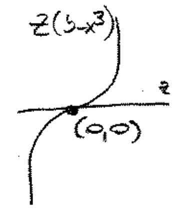
PROPERTIES (WE EXPECT):

1. $I_1 \subset I_2 \Leftrightarrow Z(I_1) \supset Z(I_2)$
2. $Z(I_1 \cap I_2) = Z(I_1) \cup Z(I_2)$
3. $Z(\sum I_J) = \bigcap_J Z(I_J)$ // 3' $\mathbb{I}(\bigcap X_J) = \sqrt{\sum \mathbb{I}(X_J)}$

RMK: The sum of two radical ideals could be not a radical ideal

EX: $I_1 = (y - x^3)$ $\quad I_2 = (y)$

$(I_1 + I_2) = (y, y - x^3) = (y, x^3)$ not radical $\quad \sqrt{(I_1 + I_2)} = (x, y)$



TOPOLOGICAL OBSERVATIONS:

① $\mathbb{A}^n$ satisfies (DCC: descending chain condition) for the closed.
i.e. $\forall$ chain $Z_1 \supset Z_2 \supset \dots \supset Z_n \supset \dots$ $\exists n_0$ s.t. $\forall n \geq n_0$ $Z_n = Z_0$

Def: A ring R is NOETHERIAN iff R satisfies ACC for the ideals $\iff \forall I_1 \subset I_2 \subset \dots \subset I_n \subset \dots$ $\exists n_0 : \forall n \geq n_0$ $I_n = I_{n_0}$

DEF: A topological space that satisfies DCC is said to be NOETHERIAN

② As all noetherian spaces, $\mathbb{A}^n$ is PARACOMPACT (not Hausdorff)
$\forall$ open cover $\mathbb{A}^n = \bigcup_{i \in I} U_i$ $\exists$ finite subset $i_1, \dots i_e \subseteq I$
s.t. $\mathbb{A}^n = \bigcup_{h=1}^{e} U_{i_h}$

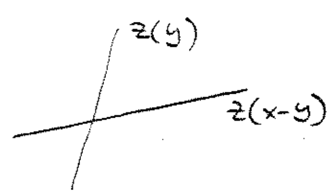
Def: Let X topological space:
(i) X is reducible if $\exists X_1, X_2$ closed $\subseteq X$: $X_1, X_2 \subsetneq X$, $(X_i \neq \phi)$ and $X = X_1 \cup X_2$
(ii) X is irreducible if it is not reducible.

EX: ① $\mathbb{R}^n$ with the Euclidean topology is Reducible
② $\mathbb{A}^n$ over an infinite field is irreducible
(if k is finite $\mathbb{A}^1$ is reducible because the Zariski topology is the discrete topology)

LEMMA: Let $X \subseteq \mathbb{A}^n$ closed $\Rightarrow$ X is irreducible iff $\mathbb{I}(X)$ is a prime idea

EX: ⓐ $\mathbb{A}^1$: the irreducible closed are the points
X irreducible iff $\mathbb{I}(x) = (x-a)$ $\Rightarrow$ $x = \{a\}$ $\exists a \in k$

ⓑ $\mathbb{A}^2$: $Z((x-y) \cdot y)$ is reducible



Pf of the LEMMA:

Suppose that $\mathbb{I}(X)$ is not prime $\Rightarrow \exists\, f, g \in R \setminus \mathbb{I}(X)$ s.t. $f \cdot g \in \mathbb{I}(X)$

$\Rightarrow X \subseteq \mathcal{Z}(f \cdot g) = \mathcal{Z}(f) \cup \mathcal{Z}(g)$.

Let $X_f = X \cap \mathcal{Z}(f) \qquad X_g = X \cap \mathcal{Z}(g)$

we want to prove that neither is empty or the whole X:

$X = X_f \cup X_g$ but neither f or g is zero for all the points in X

$\Rightarrow \quad X \supsetneq X_f$ and $X \supsetneq X_g \quad \Rightarrow X$ is REDUCIBLE!

Let $\mathbb{I}(X)$ prime and let $X = X_1 \cup X_2$ w/ $X_1 \neq X \Leftrightarrow X_1 \subsetneq X$

$\Leftrightarrow \mathbb{I}(X_1) \supsetneq \mathbb{I}(X)$.

Let us consider $f \in \mathbb{I}(X_1) \setminus \mathbb{I}(X)$.

$\forall g \in \mathbb{I}(X_2)$, $f \cdot g \in \mathbb{I}(X) \Rightarrow$ because $\mathbb{I}(X)$ is prime and $f \notin \mathbb{I}(X)$

we have $g \in \mathbb{I}(X) \Rightarrow \mathbb{I}(X_2) \subseteq \mathbb{I}(X) \Rightarrow$ they are equal $\Rightarrow X = X_2$

$(X_2 \subseteq X \Rightarrow \mathbb{I}(X) \subseteq \mathbb{I}(X_2))$ ok!

RMEMARK: X IRREDUCIBLE $\Rightarrow \mathbb{I}(X)$ PRIME HOLDS EVEN IF $K \neq \bar{K}$

EX: $\mathbb{Q}[x,y] \supseteq I$, I prime s.t. $\mathcal{Z}(I)$ is reducible.

Let $a \neq b \in \mathbb{Q}$. $f(x,y) = (x^2 - a^2)^2 + (y^2 - b^2)^2$.

Over $\mathbb{C}$ it would have been $f(x,y) = ((x^2 - a^2) + i(y^2 - b^2))((x^2 - a^2) - i(y^2 - b^2))$

but $f \in \mathbb{Q}[x,y]$ is irreducible $\Rightarrow f$ is PRIME!

HOWEVER: $\mathcal{Z}(f) = \{(a,b), (-a,b), (a,-b), (-a,-b)\}$ REDUCIBLE!

LEMMA: Let $X \subseteq \mathbb{A}^n$ closed $\Rightarrow$ X admits a finite set of irreducible components $\{X_1, \dots X_m\}$ s.t. $X = \bigcup X_i$ and the decomposition is unique up to the order.

DEF: Let X closed, $X' \subseteq X$. X' is closed of X if

⊛ X' is irreducible ⊛ $\forall$ closed $Y \supsetneq X' \Rightarrow Y$ is REDUCIBLE

EX ① X IRREDUCIBLE $\Rightarrow$ X is THE ONLY IRREDUCIBLE COMPONENT of ITSELF

② $f = x(x-1)$ $X = Z(f) \subseteq \mathbb{A}^2$

$\Rightarrow Z(x) = X_1$ $Z(x-1) = X_2$ $\Rightarrow$ X is REDUCIBLE AND X_1, X_2 are THE IRREDUCIBLE COMPONENTS (UNIQUELY DETERMINED)

ANALOGY: $\exists!$ of the factorization in $K[x_1, \dots x_n] = R$

PF (LEMMA): If X is reducible it's obvious.

Let X reducible. Let $S = \{$irreducible components of $X\}$

We want to show that $\#S < \infty$.

If $\#(S)$ was infinity:

Let $X_1 = Z_1 \cup Z_1'$ $Z_1, Z_1' \subsetneq X$ closed.

OBS: $\forall Y \in S$, Y is entirely contained in either Z_1 or Z_1' $\Rightarrow$ an infinite number of elements of S in contained in one of them, let's say Z_1 $\Rightarrow$ Z_1 is reducible.

Let us iterate the process on $Z_1 = Z_2 \cup Z_2'$ closed. We obtain $X \supsetneq Z_1 \supsetneq Z_2 \supsetneq \dots \supsetneq Z_n \supsetneq \dots$ an infinite chain of closed

$\divideontimes$ by DCC $\Rightarrow \#S < \infty$ $\Rightarrow S = \{X_1, \dots X_n\}$ and $X = X_1 \cup \dots \cup X_n$

(UNIQUENESS IS AN IMMEDIATE CONSEQUENCE of the "MAXIMALITY" of the IRREDUCIBLE COMPONENTS). OK!

EXERCISES:

1. Let X be a topological space. TRUE or FALSE (+ PROOF):

   a. If X is irreducible, then every dense subset is irreducible

   b. If X is irreducible, then every open subset, nonempty, of X is irreducible and dense in X.

   c. If every open subset nonempty of X is dense in X, then X is irreducible.

2. In $\mathbb{A}^3$, consider $X = Z(f,g)$ where $f(x,y,z) = x^2 - yz$ and $g(x,y) = xz - x$. Find the irreducible components of X.

3. With the same definitions, let us consider $\mathbb{A}^1_F$ where F is

   a. $F = \mathbb{R}$ b. $F =$ a finite field

Describe the Zariski topology on $\mathbb{A}^1_F$ and determine if $\mathbb{A}^1_F$ is irreducible in both cases.