HARTSHORNE'S ALGEBRAIC GEOMETRY - SECTION 2.1

Y.P. LEE'S CLASS

2.1.7: *Let $\mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.*

(a) *Show that $im(\phi) = \mathcal{F}/ker(\phi)$.*

Solution by Dylan Zwick

First we note that if $\mathcal{F}_1$ and $\mathcal{G}_1$ are isomorphic presheaves, then they're isomorphic on stalks, and as the stalk of a presheaf is equal to the stalk of the presheaf's sheafification, we have that $\mathcal{F}_1^+$ and $\mathcal{G}_1^+$ are isomorphic on stalks. For sheaves, isomorphic on stalks is equivalent to being isomorphic, and therefore $\mathcal{F}_1^+$ is isomorphic to $\mathcal{G}_1^+$. So, if two presheaves are isomorphic, then their respective sheafifications are isomorphic as well, as of course they must be. For any open set U we obviously have that the image $\phi(U)$ is isomorphic to $\mathcal{F}(U)/ker(\phi(U))$, as it's a homomorphism of abelian groups, and so the presheaf image of ϕ is isomorphic to the quotient presheaf $U \to \mathcal{F}(U)/ker(\phi)(U)$. As the presheafs are isomorphic, their respective sheafifications must also be isomorphic, and we have:

$$im(\phi) \cong \mathcal{F}/ker(\phi).$$

(b) *Show that $coker(\phi) = \mathcal{G}/im(\phi)$.*

Solution

To prove this we first note that if $\mathcal{G}_1 \subseteq \mathcal{F}_1$ are presheaves then for any point p we have:

$$(\mathcal{F}_1/\mathcal{G}_1)^+_p = (\mathcal{F}_1/\mathcal{G}_1)_p = ((\mathcal{F}_1)_p/(\mathcal{G}_1)_p) = ((\mathcal{F}_1^+)_p/(\mathcal{G}_1^+)_p).$$

Therefore, as they're isomorphic on stalks, we have $(\mathcal{F}_1/\mathcal{G}_1)^+ = (\mathcal{F}_1^+/\mathcal{G}_1^+)$. So, now we just note that by definition $coker(U) = \mathcal{G}(U)/\phi(U)$ and so the presheaf cokernel is ϕ is isomorphic to the presheaf quotient of $\mathcal{G}$ by the presheaf image of ϕ.

Therefore, we have that their respective sheafifications are isomorphic, and so:

$$coker(\phi) \cong \mathcal{G}/im(\phi).$$