HARTSHORNE'S ALGEBRAIC GEOMETRY - SECTION 2.1

Y.P. LEE'S CLASS

2.1.1: *Let A be an abelian group, and define the* ***constant presheaf*** *associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.*

Solution by Dylan Zwick

If we examine the constant sheaf $\mathcal{A}$ we note that for an open set $U \subseteq X$ and a continuous map $\phi : U \to A$, with A given the discrete topology, there is an obvious and natural identification between $\phi(s)$, with $s \in U$, and the corresponding element in $\mathcal{F}_s$, where $\mathcal{F}_s$ is the stalk of the constant presheaf associated to A on X. Also, we note that for any $s \in U$ there is an open set $V \subseteq U$ defined by $\phi^{-1}(\phi(s))$[1], and an element $\phi(s) \in A = \mathcal{F}(V)$, such that for all $q \in V$ we have that $\phi(s)_q = \phi(q)$, where we again use the natural identification between $\phi(s)$ and the corresponding element in $\mathcal{F}_s$. So, the constant sheaf defined in the text is the sheafification of the constant presheaf defined here.

2.1.2: (a) *For any morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, show that for each point P, $(ker\phi)_p = ker(\phi_p)$ and $(im\phi)_p = im(\phi_p)$.*

Solution by Yuchen Zhang

"$ker(\varphi)_P \subseteq ker(\varphi_P)$" Obvious.
"$ker(\varphi_P) \subseteq ker(\varphi)_P$" If $s_P \in ker(\varphi_P)$, we have $s_P \in \mathscr{F}_P$ and $\varphi_P(s_P) = 0$ in $\mathscr{G}_P$. Therefore, there exists an open set $U \ni P$ and a section $s \in \mathscr{F}(U)$ such that $s|_P = s_P$ and $\varphi(s)|_P = 0$, which means there is an open set $V \ni P$ and $\varphi(s)|_V = 0$. Hence, $\varphi(s|_V) = 0$, $s|_V \in ker(\varphi)(V)$, $s_P = (s|_V)|_P \in ker(\varphi)_P$.
"$im(\varphi)_P \subseteq im(\varphi_P)$" Obvious, since $im(\varphi)_P$ is the same

[1] We've used the discrete topology on A and the fact that ϕ is continuous

stalk of the presheaf of image before sheafification.
"$im(\varphi_P) \subseteq im(\varphi)_P$" If $s_P \in im(\varphi_P)$, we have some $t_p \in \mathscr{F}_P$ such that $\varphi_P(t_P) = s_P$. Suppose $t \in \mathscr{F}(U)$ is a section on some open neighborhood U of P such that $t|_P = t_P$. Then, $\varphi(t)|_P = \varphi_P(t_P) = s_P$, so s_P is in the stalk of the image presheaf at P. Recalling that the stalk at a point remains the same after sheafification, we have $s_P \in im(\varphi)_P$.

(b) *Show that ϕ is injective (respectively, surjective) if and only if the induced map on the stalks ϕ_p is injective (respectively, surjective) for all P.*

Solution

$$\begin{aligned}
\varphi \text{ is injective} &\iff ker(\varphi) = 0 \\
&\iff ker(\varphi)_P = 0 \qquad \forall P \in X \\
&\iff ker(\varphi_P) = 0 \qquad \forall P \in X \\
&\iff \varphi_P \text{ is injective} \quad \forall P \in X \\
\varphi \text{ is surjective} &\iff im(\varphi) = \mathscr{G} \\
&\iff im(\varphi)_P = \mathscr{G}_P \qquad \forall P \in X \quad \text{(for both sides are sheaves)} \\
&\iff im(\varphi_P) = \mathscr{G}_P \qquad \forall P \in X \\
&\iff \varphi_P \text{ is surjective} \quad \forall P \in X
\end{aligned}$$

(c) *Show that a sequence $\ldots \mathcal{F}^{i-1} \stackrel{\phi^{i-1}}{\to} \mathcal{F}^i \stackrel{\phi^i}{\to} \mathcal{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.*

Solution

$$\begin{aligned}
\text{the sequence is exact} &\iff ker(\varphi^i) = im(\varphi^{i-1}) \\
&\iff ker(\varphi^i)_P = im(\varphi^{i-1})_P \qquad \forall P \in X \\
&\iff ker(\varphi^i_P) = im(\varphi^{i-1}_P) \qquad \forall P \in X \\
&\iff \text{the stalk sequence is exact} \quad \forall P \in X
\end{aligned}$$

2.1.3: (a) *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition* *holds: for every open set $U \subset X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all i.*

Solution by Christian Martinez

We know from exercise 1.2(b) that $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective if and only if $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is surjective for all p. Thus, $\varphi : \mathscr{F} \to \mathscr{G}$ surjective implies that for each open subset $U \subset X$ and each $p \in U$ there exist an open neighborhood $V \subset X$ of p and $\tilde{s} \in \mathscr{F}(V)$ such that

$$\overline{(s, U)} = \overline{(\varphi(V)(\tilde{s}), V)} \in \mathscr{G}_p.$$

Therefore, there exists $W \subset U \cap V$ such that $p \in W$ and

$$s|_W = \varphi(V)(\tilde{s})|_W = \varphi(W)(\tilde{s}|_W).$$

Let us denote $\tilde{s}|_W$ by $t \in \mathscr{F}(W)$. Since that this is true for each $p \in U$ we get an open cover $\{U_i\}$ of U and sections $t_i \in \mathscr{F}(U_i)$ such that

$$s|_{U_i} = \varphi(U_i)(t_i), \text{ for all } i.$$

Conversely, given $\overline{(s, U)} \in \mathscr{G}_p$ (p fixed), if there exist an open cover $\{U_i\}$ of U and sections $t_i \in \mathscr{F}(U_i)$ such that

$$\varphi(t_i) = s|_{U_i} \text{ for all } i,$$

then in particular $p \in U_i$ for some i and therefore $\overline{(s, U_i)} = \overline{(s, U)} \in \mathscr{G}_p$ implying that $\varphi_p\overline{(t_i, U_i)} = \overline{(s, U)}$.

(b) *Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, and an open set U such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.*

Solution

Consider on $\mathbb{C} \setminus \{0\}$ the sheaves $\mathcal{O}$ and $\mathcal{O}^*$ given by

- $\mathcal{O}(U)$ is the additive group of holomorphic functions on U.
- $\mathcal{O}^*(U)$ is the multiplicative group of nonzero holomorphic functions on U.

Consider the map

$$\exp : \mathcal{O} \to \mathcal{O}^*$$

given by sending $f \in \mathcal{O}(U)$ to $e^{2\pi i f} \in \mathcal{O}^*(U)$. Actually, exp is a morphism of sheaves.

Note that $z \in \mathcal{O}^*(\mathbb{C} \setminus \{0\})$ is not the image of any function in $\mathcal{O}(\mathbb{C} \setminus \{0\})$ (log is not holomorphic in $\mathbb{C} \setminus \{0\}$). However, since locally there is always a branch of log, the map

$$\exp_x : \mathcal{O}_x \to \mathcal{O}_x^*$$

is surjective for each x.

2.1.6: (a) *Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Show that the natural map of $\mathcal{F}$ to the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is surjective, and has kernel $\mathcal{F}'$. Thus there is a short exact sequence:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0.$$

Solution by Dylan Zwick

The elements of the quotient sheaf will be maps, s, from U to $\cup_p \mathcal{F}''_p$, where $\mathcal{F}''(U) = \mathcal{F}(U)/\mathcal{F}''(U)$, such that:

(a) $s(p) \in \mathcal{F}''_p$ for all $p \in U$;

(b) For all $p \in U$, there exists a $V \subseteq U$ with $p \in V$ and a $t \in \mathcal{F}''(V)$ such that for all $q \in V$, $s(q) = t_q$.

The natural map from $\mathcal{F}$ to $\mathcal{F}/\mathcal{F}'$ will map any section $a \in \mathcal{F}(U)$ to the map generated by the image of a in $\mathcal{F}(U)/\mathcal{F}'(U)$.

Now, suppose we have an open set $U \subseteq X$ and a map s. Then for all $p \in U$ we have a corresponding open set $V_p \subseteq U$ and element $t(p) \in \mathcal{F}(V_p)/\mathcal{F}'(V_p)$ that satisfy the condition above. For each such $t(p)$ pick an element $t(p)' \in \mathcal{F}(V)$ that maps to $t(p)$ under the quotient. We note that this element $t(p)'$ under the natural map to $\mathcal{F}/\mathcal{F}'$ will just map to the restriction $s|_{V_p}$, and so by problem 2.1.3a) we have that the natural map is surjective.

The subsheaf $\mathcal{F}'$ is obviously contained in the kernel of the natural map. If a section $a \in \mathcal{F}(U)$ gives rise to the zero map in $\mathcal{F}(U)/\mathcal{F}'(U)$ this implies that for some open cover of U the restrictions of a to the sets in this open cover are all in $\mathcal{F}'$. This would imply that $a \in \mathcal{F}'(U)$ because $\mathcal{F}'$ is a sheaf. So, the kernel of the map is just the subsheaf $\mathcal{F}'$.

(b) *Conversely, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$, and that $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.*

Solution

If we say that α is the map $\mathcal{F}' \to \mathcal{F}$ then call the presheaf image of α G. As α is injective it induces a presheaf isomorphism $\phi : \mathcal{F}' \to G$. The map ϕ^{-1} is an injective presheaf morphism from G to the sheaf F, and so it will have a corresponding unique morphism of sheaves $\psi : im(\phi) \to F$, where $\phi^{-1} = \psi \circ \theta$, and θ is of course the sheafification map $\theta : G \to im(\phi)$, noting that $im(\phi)$ is by definition the sheafification of G. Now, according to problem 2.1.4a) this morphism ψ is injective. As ϕ^{-1} is surjective we must have that ψ is surjective as well. Therefore, ϕ is injective and surjective, and so according to problem 2.1.5 it is an isomorphism. Now, according to problem 2.1.4b) $im(\phi)$ can be identified with a subsheaf of $\mathcal{F}$, and so we have that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$.
Now, call the map $\mathcal{F} \to \mathcal{F}''$ β, then say G is the presheaf given by $U \mapsto \mathcal{F}/ker(\beta)$. Then β induces an isomorphism of presheaves $\phi : G \to \mathcal{F}''$, and we've got the situation we had before, except with ϕ and ϕ^{-1} reversed, which is just notational and doesn't matter. So, G^+ and $\mathcal{F}''$ are isomorphic, and as $G^+ = \mathcal{F}/ker(\beta)$ by definition, and as $ker(\beta) = im(\alpha) = \mathcal{F}'$ by the exactness of the sequence we have that $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$.

2.1.7: *Let $\mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.*

(a) *Show that $im(\phi) = \mathcal{F}/ker(\phi)$.*

Solution by Dylan Zwick

First we note that if $\mathcal{F}_1$ and $\mathcal{G}_1$ are isomorphic presheaves, then they're isomorphic on stalks, and as the stalk of a presheaf is equal to the stalk of the presheaf's sheafification, we have that $\mathcal{F}_1^+$ and $\mathcal{G}_1^+$ are isomorphic on stalks. For sheaves, isomorphic on stalks is equivalent to being isomorphic, and therefore $\mathcal{F}_1^+$ is isomorphic to $\mathcal{G}_1^+$. So, if two presheaves are isomorphic, then their respective sheafifications are isomorphic as well, as of course they must be. For any open set U we obviously have that the image $\phi(U)$ is isomorphic to $\mathcal{F}(U)/ker(\phi(U))$, as it's a homomorphism of abelian groups, and so the presheaf image of ϕ is isomorphic to the quotient presheaf $U \to \mathcal{F}(U)/ker(\phi)(U)$. As the presheafs are isomorphic, their respective sheafifications must also be isomorphic, and we have:

$$im(\phi) \cong \mathcal{F}/ker(\phi).$$

(b) *Show that $coker(\phi) = \mathcal{G}/im(\phi)$.*

Solution

To prove this we first note that if $\mathcal{G}_1 \subseteq \mathcal{F}_1$ are presheaves then for any point p we have:

$$(\mathcal{F}_1/\mathcal{G}_1)^+_p = (\mathcal{F}_1/\mathcal{G}_1)_p = ((\mathcal{F}_1)_p/(\mathcal{G}_1)_p) = ((\mathcal{F}_1^+)_p/(\mathcal{G}_1^+)_p).$$

Therefore, as they're isomorphic on stalks, we have $(\mathcal{F}_1/\mathcal{G}_1)^+ = (\mathcal{F}_1^+/\mathcal{G}_1^+)$. So, now we just note that by definition $coker(U) = \mathcal{G}(U)/\phi(U)$ and so the presheaf cokernel is ϕ is isomorphic to the presheaf quotient of $\mathcal{G}$ by the presheaf image of ϕ. Therefore, we have that their respective sheafifications are isomorphic, and so:

$$coker(\phi) \cong \mathcal{G}/im(\phi).$$

2.1.8: *For any open subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e., if*

$$0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \tag{1}$$

is an exact sequence of sheaves, then

$$0 \to \Gamma(U, \mathcal{F}') \xrightarrow{\phi(U)} \Gamma(U, \mathcal{F}) \xrightarrow{\psi(U)} \Gamma(U, \mathcal{F}'') \tag{2}$$

is an exact sequence of groups.

Solution by Chris Kocs

Since ϕ is injective, the induced map $\phi(U)$ must be injective for every open subset U of X, so to show that (2) is an exact sequence, we just need to show that $\ker(\phi(U))$ is equal to $\text{im}(\psi(U))$. Let $s \in \Gamma(U, \mathcal{F}')$. By problem 1.2, the induced sequence of stalks

$$0 \to \mathcal{F}'_P \xrightarrow{\phi_P} \mathcal{F}_P \xrightarrow{\psi_P} \mathcal{F}''_P \tag{3}$$

is exact as a sequence of abelian groups for every $P \in X$ since (1) is exact. So $\psi_P(\phi_P(s_P)) = 0$ for every $P \in U$. Hence, $\psi(\phi(s))_P = 0$ for every $P \in U$. (There is an open subset V of U such that the pair $<V, s|_V>$ represents the element s_P in $\mathcal{F}'_P$, the pair $<V, \phi(s)|_V>$ represents $\phi(s)_P$ in $\mathcal{F}_P$, and $<V, \psi(\phi(s))|_V>$ represents $\psi(\phi(s))_P$ in $\mathcal{F}''_P$. We must have that $\psi(\phi(\rho'_{UV}(s))) = \psi(\rho_{UV}(\phi(s))) = \rho''_{UV}(\psi(\phi(s)))$ by the definition of morphisms between sheaves, where ρ', ρ, and ρ'' are the restriction maps on $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ respectively.) For every $P \in U$ then, there is a open neighborhood V_P such that $\psi(\phi(s))|_{V_P} = 0$, so by the uniqueness condition on sheaves, $\psi(\phi(s)) = 0$. This means that $\mathrm{im}(\phi(U)) \subseteq \ker(\psi(U))$.

Now let $t \in \ker\,\psi$. For all $P \in U$, there is an $s_P \in \mathcal{F}'_P$ such that $\phi_P(s_P) = t_P$ by the exactness of (3). Thus, there exists an open covering $\{V_i\}$ of U and elements $s_i \in \mathcal{F}'$ such that $\phi(s_i) = t|_{V_i}$. Since $\phi(s_i|_{V_i \cap V_j}) = \phi(s_j|_{V_i \cap V_j}) = t|_{V_i \cap V_j}$ for all i and j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ by the injectivity of $\phi(V_i \cap V_j)$. By the existence condition on sheaves, there is an $s \in \mathcal{F}'(U)$ such that $s|_{V_i} = s_i$ for all i. By the uniqueness condition, $\phi(s) = t$, so $\ker(\psi(U)) \subseteq \mathrm{im}(\phi(U))$, concluding the proof.

2.1.9: *Direct Sum. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called the direct sum of $\mathcal{F}$ and $\mathcal{G}$, and it is denoted by $\mathcal{F} \oplus \mathcal{G}$. Show that it plays the role of direct sum and of direct product in the categorie of sheaves of abelian groups.*

Solution by Veronika Ertl

PROOF: $\mathcal{F} \oplus \mathcal{G}$ is clearly a presheaf:
(a) For every open subset $U \subseteq X$, $\mathcal{F} \oplus \mathcal{G}(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ is again an abelian group.
(b) For every inclusion $V \subseteq U$ of open subsets of X, we have the restriction map induces by the restriction maps of $\mathcal{F}$ and $\mathcal{G}$: $\rho_{UV} = (\rho^{\mathcal{F}}_{UV}, \rho^{\mathcal{G}}_{UV})$.
(0) $\mathcal{F} \oplus \mathcal{G}(\emptyset) = \mathcal{F}(\emptyset) \oplus \mathcal{G}(\emptyset) = 0 \oplus 0 = 0$.
(1) $\rho_{UU} = (\rho^{\mathcal{F}}_{UU}, \rho^{\mathcal{G}}_{UU}) = (\mathrm{Id}, \mathrm{Id}) = \mathrm{Id}$.

(2) If $W \subseteq V \subseteq U$ are three open subsets, $\rho_{UW} = (\rho^{\mathcal{F}}_{UW}, \rho^{\mathcal{G}}_{UW}) = (\rho^{\mathcal{F}}_{VW} \circ \rho^{\mathcal{F}}_{UV}, \rho^{\mathcal{G}}_{VW} \circ \rho^{\mathcal{G}}_{UV}) = \rho_{VW} \circ \rho_{UV}$.
We will show that it is indeed a sheaf:
(3) Let U be an open set in X and $\{V_i\}_{i \in I}$ a covering of U. Let $s, t \in \mathcal{F} \oplus \mathcal{G}(U)$ such that $s|_{V_i} = t|_{V_i}$ for all $i \in I$. By definition, s and t can be written in the form $s = s_{\mathcal{F}} + s_{\mathcal{G}}$ and $t = t_{\mathcal{F}} + t_{\mathcal{G}}$ with $s_{\mathcal{F}}, t_{\mathcal{F}} \in \mathcal{F}(U)$ and $s_{\mathcal{G}}, t_{\mathcal{G}} \in \mathcal{G}(U)$. This means $s_{\mathcal{F}}|_{V_i} + s_{\mathcal{G}}|_{V_i} = t_{\mathcal{F}}|V_i + t_{\mathcal{G}}|_{V_i}$ so $s_{\mathcal{F}}|_{V_i} = t_{\mathcal{F}}|_{V_i}$ and $s_{\mathcal{G}}|_{V_i} = t_{\mathcal{G}}|_{V_i}$. By hypothesis $s_{\mathcal{F}} = t_{\mathcal{F}}$ on U and $s_{\mathcal{G}} = t_{\mathcal{G}}$ on U. This shows that actually $s = t$ on U.
(4) Let $s_i \in \mathcal{F} \oplus \mathcal{G}(V_i)$ for each i, such that for each i, j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_J}$. Following the same strategy as in (3), i.e. regarding the $\mathcal{F}$ and the $\mathcal{G}$ components of all s_i, we find for each component an element over U that satisfies (4) of Hartshorne. Matching these together, we find $s \in \mathcal{F} \oplus \mathcal{G}(U)$ such that $s|_{V_i} = s_i$ for all $i \in I$.

We have seen, that the direct sum as defined is again in the category of sheaves over X. In particular, it is compatible with the morphisms as defined in the lecture. The YONEDA-Lemma shows then, that the direct sum is uniquely defined in this category (and so is the direct product).

2.1.11: Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathcal{F}_i) = \varinjlim \Gamma(X, \mathcal{F}_i)$.

Solution by Dylan Zwick

Take an open set $U \subseteq X$ and an element $s \in \varinjlim \mathcal{F}_i(U)$. This element will have a representative $t_s \in \mathcal{F}_s(U)$, where $\mathcal{F}_s$ is a scheme in our direct system. Now, take an open cover of U by sets V_i, and note that as X is noetherian it is compact, so we may assume open cover is finite. Say there are N open sets in this finite cover. For each of these open sets find a representative $t_{s|_{V_i}} \in \mathcal{F}_{s|_{V_i}}$. This gives us a finite set of schemes in our direct system $S = \{\mathcal{F}_s(V), \mathcal{F}_{s|_{V_1}}, \ldots, \mathcal{F}_{s|_{V_N}}\}$, and so we can find a scheme $\mathcal{F}$ for which every scheme in S is a subscheme. Each of our representative elements $t_s, t_{s|_{V_1}}, \ldots, t_{s|_{V_N}}$ will have a corresponding representative in the appropriate abelian group in$\mathcal{F}$. Call these representatives $t \in \mathcal{F}(U), t|_{V_1} \in \mathcal{F}(V_1), \ldots, t|_{V_N} \in \mathcal{F}(V_N)$. Now, we note that if $s|_{V_i} = 0$ for each V_i then the corresponding representatives $t|_{V_i} = 0$. As $\mathcal{F}$ is a scheme, this implies that $t = 0$, which means $s = 0$. So, the presheaf satisfies the first requirement to be a sheaf.

As for the second requirement, suppose we have an open set $U \subseteq X$ and an open cover of U by sets V_i (which we may again assume is a finite open cover) such that for each V_i we have an element $s_i \in \varinjlim(V_i)$, and for any two of these open sets if $V_i \cap V_j \neq \emptyset$ then $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. We note, of course, that there are only a finite number of possible pairs of sets, as there is a finite number of such sets. For every element in an open set described here take a representative in an appropriate scheme $\mathcal{F}_i$. Find a scheme $\mathcal{F}$ that is larger than any of these schemes, and examine the corresponding representatives in this case. We note that as $\mathcal{F}$ is a scheme we must have an element $t \in \mathcal{F}(U)$ that restricts to the appropriate corresponding representative on each V_i. This element t then has a corresponding element $s \in \varinjlim(\mathcal{F}_i(U))$ that satisfies $s|_{V_i} = s_i$. So, the presheaf satisfies the second requirement to be a sheaf.

The presheaf satisfies both requirements to be a sheaf, and is therefore a sheaf. I note that, if this proof is correct, then the requirement that X be a noetherian topological space is stronger than it needs to be. We need only require that X is compact.

2.1.13: *Espace Étale of a Presheaf. (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [1. Ch. II, Section 1.2].) Given a presheaf $\mathcal{F}$ on X, we define a topological space $Spe(\mathcal{F})$, called the espace étalé of $\mathcal{F}$, as follows. As a set, $Spe(\mathcal{F}) = \cup_{p \in X} \mathcal{F}_p$. We define a projection map $\pi : Sp(\mathcal{F}) \to X$ by sending $s \in \mathcal{F}_p$ to P. For each open set $U \subseteq X$ and each section $s \in \mathcal{F}(U)$, we obtain a map $\overline{s} : U \to Sp(\mathcal{F})$ by sending $P \mapsto s_p$, its germ at P. This map has the property that $\pi \circ \overline{s} = id_U$, in other words, it is a "section" of π over U. We now make $Sp(\mathcal{F})$ into a topological space by giving it the strongest topology such that all the maps*

$\overline{s} : U \to Sp(\mathcal{F})$ for all U, and all $s \in \mathcal{F}(U)$, are continuous. Now show that the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ can be described as follows: for any open set $U \subseteq X, \mathcal{F}^+(U)$ is the set of continuous sections of $Sp(\mathcal{F})$ over U. In particular, the original presheaf $\mathcal{F}$ was a sheaf if and only if for each U, $\mathcal{F}(U)$ is equal to the set of all continuous sections of $Sp(\mathcal{F})$ over U.

Solution by Yuchen Zhang

The first statement follows immediately from the definition. Recalling that the presheaf $\mathscr{F}$ is a sheaf if and only if $\mathscr{F}(U) = \mathscr{F}^+(U)$ for any open set U. The second statement follows obviously.

2.1.14: *Support. Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open subset U. The support of s, denoted by Supp s, is defined to be $\{P \in U \ : \ s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that Supp s is a closed subset of U.*

Solution by Christian Martinez

Let $A = \{p \in U \ : \ s_p = 0\}$. If $p \in A$ then $\overline{(s, U)} = 0 \in \mathscr{F}_p$, i.e, there exists an open neighborhood $V \subset U$ of p such that $s|_V = 0$. Let $q \in V$, since $V \subset U$ and $s|_V = 0$ then $s_q = \overline{(s, U)} = 0 \in \mathscr{F}_q$. Thus, $V \subset A$ and A is open.

2.1.16: - *Flasque Sheaves. A sheaf $\mathcal{F}$ on a topological space X is flasque is for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.*

(a) *Show that a constant sheaf on an irreducible topological space is flasque.*

Solution by Dylan Zwick

Here take $\mathcal{F}$ to be the constant sheaf for an abelian group A on an irreducible topological space X. Take any open set $V \subseteq X$ and a continuous function $f : V \to A$. Take some element $a \in f(V)$, and look at the subset $f^{-1}(a) \subseteq V$. As we've given A the discrete topology, the element $a \in A$ is both open and closed, and so therefore both $f^{-1}(a)$

and $f^{-1}(a)^c$ are closed in V. This implies they are both restrictions of closed sets in X, call them $X(f^{-1}(a))$ and $X(f^{-1}(a)^c)$. We now note that $X(f^{-1}(a)) \cup (X(f^{-1}(a)^c) \cup V^c) = X$, and as X is irreducible, this implies one must be equal to X. As $f^{-1}(a)$ is necessarily nonempty, this implies $(X(f^{-1}(a)^c) \cup V^c) \neq X$, which requires $X(f^{-1}(a)) = X$, which implies $f^{-1}(a) = V$. Therefore, the map f must be the *constant* map $f(V) = a$, which is obviously a restriction of the same constant map on any set U, $V \subseteq U$. So, for any $V \subseteq U$ we have $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective, and therefore $\mathcal{F}$ is flasque.

(b) *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathcal{F}'$ *is flasque, then for any open set* U*, the sequence* $0\mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ *of abelian groups is also exact.*

Solution

We proved in problem 2.1.8 that the functor $\Gamma(U, \cdot)$ is left exact, and so the only thing we need to prove here (that is, the only thing that requires $\mathcal{F}'$ to be flasque, and which isn't true for any exact sequence of sheaves) is that the map $\mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective. This is a little harder than it might look.

Take an element $s \in \mathcal{F}'(U)$ and a point $p \in U$. Map s to $s_p \in \mathcal{F}''_p$. As the sequence of stalks is exact there is an open set $V_p \subseteq U$ and a section $t_p \in \mathcal{F}(V_p)$ such that $g(t_p) = s|_{V_p}$, where g is the map from $\mathcal{F}$ to $\mathcal{F}''$ in our sequence. Now, take some other point $q \in U$, and its corresponding $t_q in \mathcal{F}(V_q)$ constructed in the same manner as above. If $V_p \cap V_q = \emptyset$ then there will be nothing to prove. If $V_p \cap V_q \neq \emptyset$, then we note that $t_p|_{V_p \cap V_q} - t_q|_{V_p \cap V_q} \in ker(g)$, as both elements in the difference map to $s_{V_p \cap V_q} \in \mathcal{F}''(V_p \cap V_q)$. So, as our sequence is right exact on the open set $V_p \cap V_q$ we know that $t_p|_{V_p \cap V_q} - t_q|_{V_p \cap V_q} = f(r)$, where f is the map from $\mathcal{F}'$ to $\mathcal{F}$. Now, as $\mathcal{F}'$ is flasque, we know there is an element $\tilde{r} \in \mathcal{F}'(V_q)$ such that $\tilde{r}|_{V_p \cap V_q} = r$. Finally, we note that $g(t_q - f(\tilde{r})) = g(t_q)$, and so $t_q - f(\tilde{r})$ is an equally valid representative in the set V_q, where by equally valid we mean that its image under g is s_{V_q}. So, at the end of the day what this means is that for any two open sets $V_1, V_2 \subseteq U$

if there exists sections $t_1 \in \mathcal{F}(V_1$ and $t_2 \in \mathcal{F}(V_2)$ such that $g(t_1) = s|_{V_1}$ and $g(t_2) = S|_{V_2}$ then there exist sections of $\mathcal{F}(V_1)$ and $\mathcal{F}(V_2)$ satisfying these requirements that agree on the overlap $V_1 \cap V_2$.
Now, suppose we have a collection of open sets $V_\alpha \subseteq U$ and corresponding sections $t_\alpha \in \mathcal{F}(V_\alpha)$ such that $g(t_\alpha) = s|_{V_\alpha}$ for all α and the sections for any two open sets agree on the overlap. Then if we say $V = \cup V_\alpha$ then there is an element $t \in \mathcal{F}(V)$ such that $g(t) = s|_V$. As $\mathcal{F}$ is a sheaf we know that there is an element $t \in \mathcal{F}(V)$ such that $t|_{V_\alpha} = t_\alpha$ for all α. Given $g(t_\alpha) = s|_{V_\alpha}$ for all α we know that $g(t)$ is equal to $s|_V$ locally, and therefore as $\mathcal{F}''$ is a sheaf $g(t) = S|_V$.
Now we're almost done. We can take an open cover of U using open sets of the form V_p described in the first paragraph. Call a consistent subset of these sets a subset such that we can choose our sections such that they agree on the overlap. We know that the set of consistent subsets is nonempty, as it must include every set V_p alone. Suppose we have a consistent subset V_β and a subset $W \in V_\alpha$ such that $V_\beta \cup W$ is not consistent. Then, as demonstrated in paragraph 2, we can construct from V_β its union V with corresponding section, and we know from paragraph 1 that we can choose a section of W such that it's consistent with V. This is a contradiction, and therefore no such open set W exists. Therefore, we can completely cover U with consistent subsets, and therefore we can find an element $t \in U$ such that $f(t) = s$. Done!

(c) *If* $0\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathcal{F}'$ *and* $\mathcal{F}$ *are flasque, then* $\mathcal{F}''$ *is flasque.*

Solution

Take an pair of open sets $V \subseteq U$ and a section $s \in \mathcal{F}''(V)$. We know from the previous exercise that the sequence on V is exact, and so there must be an element $t \in \mathcal{F}(V)$ such that $f(t) = s$. As $\mathcal{F}$ is flasque, there is a corresponding element $r \in \mathcal{F}(U)$ such that $r|_V = t$. This element r will map to an element $f(r) \in \mathcal{F}''(U)$, and as the diagram:

must commute we know that that $f(r)|_V = s$. Therefore, $\mathcal{F}''$ is flasque.

(d) *If $f : X \to Y$ is a continuous map, and if $\mathcal{F}$ is a flasque sheaf on X then $f_*\mathcal{F}$ is a flasque sheaf on Y.*

Solution

This is pretty trivial. Take a pair of open sets $V \subseteq U$ in Y, and a section $s \in f_*\mathcal{F}(V)$. This s corresponds exactly, by definition, to an element $s \in \mathcal{F}(f^{-1}(V))$, and as $\mathcal{F}$ is flasque we know there is an element $t \in \mathcal{F}(f^{-1}(U))$ such that $t|_{f^{-1}(V)} = s$. This element t corresponds exactly with an element $t \in f_*\mathcal{F}(U)$. I've allowed some abuse of notation in refering to sections of $f_*\mathcal{F}$ with the same letters as sections of $\mathcal{F}$, but I've done so to indicate the the sheaves are, by definition, essentially the same, and so have the same sections.

(e) *Let $\mathcal{F}$ be any sheaf on X. We define a new sheaf $\mathcal{G}$, called the sheaf of discontinuous sections of $\mathcal{F}$ as follows. For each open set $U \subseteq X$, $\mathcal{G}(U)$ is the set of maps $s : U \to \bigcup_{p \in U} \mathcal{F}_p$ such that for each $p \in U$, $s(P) \in \mathcal{F}_p$. Show that $\mathcal{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathcal{F}$ to $\mathcal{G}$.*

Solution

This is also pretty obvious. For any pair of open sets $V \subseteq U$ and any section $s \in \mathcal{G}(V)$ just define a section $t \in \mathcal{G}(U)$ that is equal to s on V and is zero outside of V. This t is a discontinuous section on U, so it's a perfectly valid element of $\mathcal{G}(U)$, and it obviously restricts to s on V.
Any element $s \in \mathcal{F}(U)$ satisfies the requirement of a continuous section, and therefore also satisfies the less restrictive requirement of a discontinuous section, and so therefore represents a discontinuous section on U. The natural injective morphism is obvious.

2.1.17: ***Skyscraper sheaves.*** *Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify* *that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$ denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf". Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i : \{P\}^- \to X$ is the inclusion.*

Solution by Stefano Urbinati

Let $Q \in \{P\}^-$. Then for every open set $Q \in V$, we have that $P \in V$ and $i_P(A)(V) = A$ and defining the stalk we have a direct limit of a constant group, that is $i_P(A)_Q = A$. If $Q \notin \{P\}^-$ then there exists an open set $Q \in W$ such that $P \notin W$, that is, for any section $s : W \to i_P(A)(W)$, then $s \equiv 0$, that implies that $s_Q = 0$.
By definition we have that $i_*(A)(U) = A(i^{-1}(U))$; now, if $P \in U$ we have that $i^{-1}(U) = \{P\}^-$ and $A(\{P\}^-) = A$ by definition. If $P \notin U$, then $i^{-1}(U) = \emptyset$ and $A(\emptyset) = 0$.

2.1.18: ***Adjoint Property of*** f^{-1}. *Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, and for any sheaf $\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y,*

$$Hom_X(f^{-1}\mathcal{G}, \mathcal{F}) = Hom_Y(\mathcal{G}, f_*\mathcal{F}).$$

Hence we say that f^{-1} is a left adjoint of f_* and that f_* is a right adjoint of f^{-1}.

Solution by Dylan Zwick

The map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ will take any open subset $U \subseteq X$ and associate with it the abelian group given by:

$$f^{-1}f_*\mathcal{F}(U) = \lim_{V \subseteq f(U)} f_*\mathcal{F}(V) = \lim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)).$$

In words, this will be the group given by all elements that are images of elements in larger open sets $W \subseteq X$ where $U \subseteq W$ and W is the preimage of an open set in Y.

To understand the map $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ we take a look at the map $f_* f^{-1}\mathcal{G} \to \mathcal{G}$. This map will take any open subset $V \subseteq Y$ and associate with it the abelian group given by:

$$f_* f^{-1}\mathcal{G}(V) = (f^{-1}\mathcal{G})(f^{-1}(V)) = \lim_{W \subseteq f(f^{-1}(V))} \mathcal{G}(W) = G(W).$$

So, it's the identity, and therefore its inverse is also the identity. Thus, the map $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is the identity morphism.

To prove the bijection of sets:

$$Hom_X(f^{-1}\mathcal{G}, \mathcal{F}) = Hom_Y(\mathcal{G}, f_*\mathcal{F})$$

we first assume we've got an element $\phi \in Hom_X(f^{-1}\mathcal{G}, \mathcal{F})$ and show how to find a corresponding element in $Hom_Y(\mathcal{G}, f_*\mathcal{F})$. Suppose we have an open set $V \subseteq Y$ and an element $s \in \mathcal{G}(V)$. This element s corresponds in an obvious way with a unique element $s_1 \in (f^{-1}\mathcal{G})(f^{-1}(V))$, and so $\phi(s_1) \in \mathcal{F}(f^{-1}(V))$. This will again correspond in a (very) obvious way with an element $\tilde{\phi(s_1)} \in (f_*\mathcal{F})(V)$. So, our map ϕ induces a unique element in $Hom_Y(\mathcal{G}, f_*\mathcal{F})$. Call this map of homomorphisms ρ.

Going the other way, assume we've got an element $\psi \in Hom_Y(\mathcal{G}, f_*\mathcal{F})$. Take an open subset $U \subseteq X$, and an element $t \in (f^{-1}\mathcal{G})(U)$. So, $t \in \lim_{V \supseteq f(U)} \mathcal{G}(V)$. Take a representative $t_1 \in \mathcal{G}(V_1)$. Then $\psi(t_1) \in f_*\mathcal{F}(V_1)$, which corresponds uniquely to an element $\tilde{\psi(t_1)} \in \mathcal{F}(f^{-1}(V_1))$. If we then restrict this element $\tilde{\psi(t_1)}$ to $U \subseteq f^{-1}(V_1)$ we have a well defined map. We note that for any other representative of t, say t_2, we'd have to have that t_1 and t_2 restrict to the same element in an open subset containing $f(U)$, and so the restriction of $\tilde{\psi(t_2)}$ to U would have to be the same as the restriction of $\tilde{\psi(t_1)}$, and so the map is indeed well defined. Call this map of homomorphisms σ.

Finally, we need to prove that the two mappings defined here compose to form the identity map. But this is easy. Take a map $\psi \in Hom_Y(\mathcal{G}, f_*\mathcal{F})$ and an open set $V \subseteq Y$. The map $\rho(\sigma(\psi))$ will take an element $s \in \mathcal{G}(V)$ to the unique corresponding element $s_1 \in (f^{-1}\mathcal{G})(f^{-1}(V))$, which will then be taken by $\sigma(\psi)$ to the obvious representative $s \in \mathcal{G}(V)$, which will then be taken by ψ to $\psi(s)$. So, $\rho(\sigma(\psi)) = \psi$ and therefore the composition of the maps is the identity. So, we've got our bijection.

2.1.19: - ***Extending a Sheaf by Zero.*** *Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be the inclusion.*

(a) *Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_*(F))_P$ of the direct image sheaf on X is $\mathcal{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence, we call $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathcal{F}$ instead of $i_*\mathcal{F}$, and say "consider $\mathcal{F}$ as a sheaf on X" when we mean "consider $i_*\mathcal{F}$."*

Solution by Chris Kocs

If $P \in Z$, then

$$(i_*\mathcal{F})_P = \varinjlim_V (i_*\mathcal{F})(V) = \varinjlim_V \mathcal{F}(i^{-1}(V)) = \varinjlim_V \mathcal{F}(V \cap Z) = \mathcal{F}_P,$$

where the direct limit is taken over all open sets V containing P in X. Suppose $P \notin Z$. For any pair $<V, s>$ representing an element in $(i_*(\mathcal{F}|_Z))_P$ where V is an open subset of X and $s \in (i_*(\mathcal{F}|_Z))(V)$, $s|_{V\cap U} = 0$ as $(i_*(\mathcal{F}|_Z))(V \cap U) = \mathcal{F}(i^{-1}(\emptyset)) = 0$. Hence, $(i_*(\mathcal{F}|_Z))_P = 0$.

(b) *Now let $\mathcal{F}$ be a sheaf on U. Let $j_!\mathcal{F}$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!\mathcal{F})_P$ is equal to $\mathcal{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathcal{F}$ is the only sheaf on X which has this property and whose restriction to U is $\mathcal{F}$. We call $j_!\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside U.*

Solution

Denote by $\mathcal{F}'$ the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. If $P \in U$,

$$(j_!\mathcal{F})_P = \mathcal{F}'_P = \varinjlim_V \mathcal{F}'(V) = \varinjlim_V \mathcal{F}'(V \cap U) = \varinjlim_V \mathcal{F}(V \cap U) = \mathcal{F}_P,$$

where the direct limit is taken over all open sets V containing P in X. If $P \notin U$, then $\mathcal{F}'(V) = 0$ for all open neighborhoods V of P in X. Hence, $(j_!\mathcal{F})_P = \mathcal{F}'_P = 0$.
Now, the morphism $\theta : \mathcal{F}' \to j_!\mathcal{F}$ associated with the presheaf $\mathcal{F}'$ restricts to a morphism $\mathcal{F}'|_U \to j_!\mathcal{F}|_U$. Recall from the proof of sheafification that θ_P is an isomorphism

for all $P \in X$, so by problem 1.2, $\mathcal{F}'|_U = \mathcal{F}$ is isomorphic to $j_!\mathcal{F}|_U$ as sheaves on U. Suppose $\mathcal{G}$ is another sheaf who restriction to U is $\mathcal{F}$ and whose stalk $\mathcal{G}_P$ is $\mathcal{F}_P$ if $P \in U$ and 0 if $P \notin U$. There is a morphism $\phi : \mathcal{F}' \to \mathcal{G}$ sending every open subset V of X to $\mathcal{F}(V)$ if $V \subseteq U$, 0 otherwise. Moreover, ϕ factors through θ. Since ϕ_P is an isomorphism for all $P \in X$ by construction, $j_!\mathcal{F}|_U$ is isomorphic to $\mathcal{G}$.

(c) *Now let $\mathcal{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,*

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0$$

Solution

Let θ and ϕ be defined as in part (b) except using $\mathcal{F}|_U$ instead of $\mathcal{F}$ and $\mathcal{F}$ instead of $\mathcal{G}$. Then there exists a morphism $\phi' : j_!(\mathcal{F}|_U) \to \mathcal{F}$ such that $\phi = \phi' \circ \theta$. Consider the sequence of sheaves

$$0 \to j_!(\mathcal{F}|_U) \xrightarrow{\phi'} \mathcal{F} \xrightarrow{\psi} i_*(\mathcal{F}|_Z) \to 0, \tag{4}$$

where $\psi(\Gamma(V, \mathcal{F})) = \Gamma(V \cap Z, \mathcal{F})$ for all open subsets V of X. For $P \in X$, consider the corresponding sequence of stalks

$$0 \to (j_!(\mathcal{F}|_U))_P \xrightarrow{\phi'_P} \mathcal{F}_P \xrightarrow{\psi_P} (i_*(\mathcal{F}|_Z))_P \to 0.$$

If $P \in U$, then by (a) and (b), the above sequence becomes

$$0 \to \mathcal{F}_P \xrightarrow{\phi'_P} \mathcal{F}_P \xrightarrow{\psi_P} 0 \to 0,$$

where ϕ'_P is an isomorphism. If $P \notin U$, then we have the sequence

$$0 \to 0 \xrightarrow{\phi'_P} \mathcal{F}_P \xrightarrow{\psi_P} \mathcal{F}_P \to 0,$$

where ψ_P is an isomorphism. Therefore, in either case, the sequence is exact, so by problem 1.2, (4) is exact as a sequence of sheaves.

2.1.20: - ***Subsheaf with Supports***. *Let Z be a closed subset of X, and le $\mathcal{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathcal{F})$ to be the subgrup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support is contained in Z (recall:* $\text{Supp} s = \{P \in X | s_P \neq 0\}$*).*

(a) *Show that the presheaf* $V \mapsto \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$ *is a sheaf. It is called the subsheaf of* $\mathcal{F}$ *with supports in* Z*, and it is denoted by* $\mathcal{H}^0_Z(\mathcal{F})$.

Solution by Veronika Ertl

Again, (0),(1) and (2) are easy to verify, as it comes from a sheaf $\mathcal{F}$. We have to verify the uniqueness and the glueing property.
(3) Let U be an open set,and $\{V_i\}$ a covering of U. Let $s, t \in \Gamma_{Z\cap U}(U, \mathcal{F}|_U)$, such that $s|_{V_i} = t|_{V_i}$. The sections s and t are in fact in $\mathcal{F}(U)$ with the additional property, that their support is in $U \cap Z$, and we can just apply the uniqueness property of the original sheaf.
(4) The same reasoning holds for the glueing property. Let $s_i \in \Gamma_{Z\cap V_i}(V_i, \mathcal{F}|_{V_i})$ such that they coincide on the overlaps. As they are in the $\mathcal{F}(V_i)$ respectively, there is $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i. We have to check that the support of this s is in $Z \cap U$. Let $P \in \operatorname{Supp} s$. As $P \in U$, there is i, such that $P \in V_i$ and therefore $s_P = s_{iP}$. This means, that $P \in \operatorname{Supp} s_i$ which is by definition contained in $Z \cap V_i \subset Z \cap U$.
However, I think (but I'm not sure), it should contain the zero section by definition for every open subset?!?

(b) *Let* $U = X - Z$*, and let* $j : U \to X$ *be the inclusion. Show there is an exact sequence of sheaves on* X

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U).$$

Furthermore, if $\mathcal{F}$ *is flasque, the map* $\mathcal{F} \to j_*(\mathcal{F}|_U)$ *is surjective.*

Solution

Consider the following sequence:

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} j_*(\mathcal{F}|_U).$$

Recall that for every open subset $V \subseteq X$, the group $\mathcal{H}^0_Z(\mathcal{F})(V)$ is the subgroup of $\mathcal{F}(V)$ consisting of all sections, whose support is in $Z \cap V$. So the morphism α is given on open subsets by inclusions

$$\alpha(V) : \mathcal{H}^0_Z(\mathcal{F})(V) \to \mathcal{F}(V)\,, \quad s \mapsto s,$$

and therefore it is clearly injective (for injectivity it is sufficient to check it on open subsets). The sheaf $j_*(\mathcal{F}|_U)$ is given on open subsets as

$$V \mapsto j_*(\mathcal{F}|_U)(V) = \mathcal{F}|_U(j^{-1}(V)) = \mathcal{F}|_U(U \cap V).$$

It follows, that the morphism $\beta : \mathcal{F} \to j_*(\mathcal{F}|_U)$ is given on open subsets by

$$\beta(V) : \mathcal{F}(V) \to j_*(\mathcal{F}|_U)(V), \quad s \mapsto s|_{U\cap V}.$$

Now we have to check that $\operatorname{Im}\alpha = \operatorname{Ker}\beta$. As we deal with sheaves associated to presheaves (at least in the case of the image), we have to check it on stalks. We consider two cases: $P \in U$ and $P \in Z$.

Case 1: $P \in Z$: It is not difficult to determine the image of α_P. Let $s_p \in \mathcal{H}^0_Z(\mathcal{F})_P$ represented by $\langle s \in \mathcal{H}^0_Z(\mathcal{F})(V), V\rangle$, where V is an open subset containing P ans $s \in \mathcal{F}(V)$ such that $\operatorname{Supp} s \subseteq Z \cap V$. Since P is in Z, we may choose V containing P small, and see, that $\Im(\alpha_P) = \mathcal{F}_P$ (it contains the zero section by definition, cf. **(a)**).

And indeed, $\mathcal{F}_P$ is the kernel of β_P: Let $s_P \in \mathcal{F}_P$, represented by $\langle s, V\rangle$. Since P is not in U, and $\beta(V)$ is the restriction to $V \cap U$, $\beta_P(s_p) = 0$.

Case 2: $P \in U$: To determine the image of α_P, note that $P \notin Z$. If $s_P \in \mathcal{H}^0_Z(\mathcal{F})_P$ it can be represented by $\langle s, V\rangle$, where s is a section of$\mathcal{F}(V)$ such that $\operatorname{Supp} s \subseteq Z \cap V$. As we take the direct limit of $Z \cap V$ over all V containing P, this is the empty set. Thus, we get $s_P = 0$. So $\Im(\alpha_P) = 0$. To determine the kernel of β_P let again s_P be represented by $\langle s, V\rangle$. Since $\beta(V)$ is the restriction to $V \cap U$ and $P \in U$, we can choose $V \subseteq U$, and we see, that s_p mustbe zero, if it's in the kernel of β_P. This shows the first part of the assertion.

Let now $\mathcal{F}$ be flasque. This means, that for every inclusion $V \subseteq U$ of open subsets, the restriction map is surjective. To show that $\mathcal{F} \to J_*(\mathcal{F}|_U)$ is surjective, we verify this again on stalks. Let $t_P \in j_*(\mathcal{F}|_U)_P$ repesented by $\langle t \in \mathcal{F}|_U(U \cap V), V\rangle$, $V \ni P$. If $P \in U$, we can choose V such that $P \in V \subseteq U$. This means that $\mathcal{F}|_U(U \cap V) = \mathcal{F}(V)$ and so $t_P \in \mathcal{F}_P$. In this case, β_P is just the identity and naturally surjective. If $P \in Z$, $\beta(V)$ is the restriction map $\mathcal{F}(V) \to \mathcal{F}|_U(U \cap V) = \mathcal{F}(U \cap V)$. But since $\mathcal{F}$ is flasque, this map is surjective for every V, so t has a preimage in $\mathcal{F}(V)$. Choosing V small enough, we see, that t_P has a preimage in $\mathcal{F}_P$. So, β is surjective.

2.1.22: *Gluing Sheaves*. *Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j an isomorphism $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ such that*

(1) *for each i, $\phi_{ii} = id$, and*
(2) *for each i, j, k, $\phi_{ik} = \phi_{jk}\ \phi_{ij}$ on $U_i \cap U_j \cap U_k$.*

Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that for each $i, j \psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathcal{F}$ is obtained by gluing the sheaves $\mathcal{F}_i$ via the isomorphisms ϕ_{ij}.

Solution by Ray Lai

For $X = \cup U_i$ with sheaves $\mathcal{F}_i$ on U_i, we define the presheaf $\mathcal{F}$ by

$$\mathcal{F}(U) = \left\{ \begin{array}{ccc} 0 & \text{if} & U \nsubseteq U_i \ for\ any\ i. \\ \mathcal{F}_i(U) & \text{if} & U \subseteq U_i \ for\ some\ i. \end{array} \right\}$$

Then from properties of isomorphisms ϕ_{ij}, it's easy to show this presheaf is well-defined. Note however this may NOT be a sheaf. In order to get the required sheaf, we take the sheafification and still denote it by $\mathcal{F}$. The natural map θ between the original presheaf and its sheafification gives canonical isomorphisms $\theta_i : \mathcal{F}(U_i) \to \mathcal{F}_i(U)$.