Math 2280 - Lecture 9

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In the last two lectures we've talked about differential equations for modeling populations. Today we'll return to the theme we touched upon in our second lecture, acceleration-velocity models, and see how differential equations can be used to model air resistance. We'll also discuss concepts with cool names, like "terminal velocity" and "escape velocity".

The problems for this section are:

Section 2.3 - 1, 2, 4, 10, 24

Acceleration-Velocity Models

If an object is close to the earth's surface and we neglect any effects from air resistance, then the object will experience a constant downward force from gravity, and Newton's second law tells us that:

$$m\frac{dv}{dt} = F_G$$

where m is the object's mass, v is the object's velocity, and F_G is the constant force from gravity, which will be $-mg$.

This is very simple, and can be an OK model for some correspondingly simple physical situations, but in almost every real life situation we'll need to take air resistance into account. The phenomenon of air resistance is a pretty complicated one, which we won't discuss in depth, but we will look at some air resistance models that give us a better idea of what goes on when an object falls on the earth's surface, and will also demonstrate mathematically some important behaviors we observe in real life.

Resistance Proportional to Velocity

The first model we'll consider is the situation where air resistance is proportional to velocity, and in the opposite direction:

$$F_R = -kv$$

where k is a positive constant, and v is the object's velocity. Now, combining the air resistance with the (still assumed to be constant) force from gravity and again using Newton's second law we get:

$$m\frac{dv}{dt} = -kv - mg$$
$$\Rightarrow \frac{dv}{dt} = -\rho v - g$$

where $\rho = \dfrac{k}{m}$.

Example - Solve the first-order differential equation given above.

Solution - We can rewrite this differential equation as:

$$\frac{dv}{dt} + \rho v = -g.$$

The integrating factor for this differential equation is $\sigma(t) = e^{\int \rho dt} = e^{\rho t}$.[1] If we multiply both sides by $\sigma(t)$ we get:

$$e^{\rho t}\frac{dv}{dt} + \rho e^{\rho t} v = -g e^{\rho t}.$$

We can rewrite this as:

$$\frac{d}{dt}(e^{\rho t} v) = -g e^{\rho t}.$$

[1]The integrating factor is usually represented with the letter ρ, but here I'm using σ because we've already used ρ.

Integrating both sides with respect to t gives us:

$$e^{\rho t} v = -\frac{g}{\rho} e^{\rho t} + C.$$

Solving this for v we get:

$$v(t) = Ce^{-\rho t} - \frac{g}{\rho}.$$

If we set $v(0) = v_0$ then we have $C = v_0 + \frac{g}{\rho}$, and so our final solution is:

$$v(t) = \left(v_0 + \frac{g}{\rho} \right) e^{-\rho t} - \frac{g}{\rho}.$$

We note as $t \rightarrow \infty$ our velocity approaches the value $-\frac{g}{\rho}$. This is called the object's *terminal velocity*. The absolute value of this is called the object's *terminal speed* and is given by:

$$|v_{\tau}| = \frac{mg}{k}.$$

This phenomenon of terminal speed is what makes skydiving possible.

Example - A woman bails out of an airplane at an altitude of 10,000 ft, falls freely for 20s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance $\rho v \ ft/s^2$, taking $\rho = .15$ without the parachute and $\rho = 1.5$ with the parachute.

Solution - We have:

$$v(t) = \left(v_0 + \frac{g}{\rho} \right) e^{-\rho t} - \frac{g}{\rho}.$$

If we integrate this to find $x(t)$ we get:

$$x(t) = -\frac{g}{\rho}t - \frac{1}{\rho}\left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} + C.$$

Plugging in the initial condition $x(0) = x_0$ we get:

$$C = x_0 + \frac{1}{\rho}\left(v_0 + \frac{g}{\rho}\right) = x_0 + \frac{1}{\rho}(v_0 - v_\tau).$$

Using this value for C after a little algebra our equation for $x(t)$ becomes:

$$x(t) = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}).$$

Now, the initial distance is $x_0 = 10,000$ ft, the initial velocity is $v_0 = 0$ ft/s, the terminal velocity is $v_\tau = -\dfrac{32.2}{.15}$ ft/s, and $\rho = .15$/s. The total distance traveled in the first 20 seconds is:[2]

$$x(20) = 10,000 - \left(\frac{32.2}{.15}\right)(20) = \frac{1}{.15}\left(0 + \frac{32.2}{.15}\right)(1 - e^{-.15(20)}) \approx 7,067ft.$$

The velocity of the skydiver after 20 seconds is:

$$v(20) = \left(0 + \frac{32.2}{.15}\right)e^{-.15(20)} - \frac{32.2}{.15} \approx 204 \text{ ft/s}.$$

Now, to find the total time for the rest of the trip down we want to solve for t_f in the equation:

$$0 = 7,067 - \left(\frac{32.2}{1.5}\right)t_f + \left(\frac{1}{1.5}\right)\left(-204 + \frac{32.2}{1.5}\right)(1 - e^{-1.5t_f}).$$

Using a calculator to find this we get:

$$t_f \approx 340 \text{ seconds}.$$

So, the total skydive time is about $340s + 20s = 360s$, or about 6 minutes.

[2]Leaving out units on the intermediate steps. Trust me, they work out.

Variable Gravitational Acceleration

The model of constant gravitation only works when we're close to the surface of the earth, and the distances we're dealing with are small relative to the radius of the earth. If we start to deal with larger distances, then we must take into account that acceleration from gravity is weaker the farther we are away from the earth. Newton's law of universal gravitation tells us that the force from gravity experienced a distance r from the center of the earth will be:

$$F = -\frac{GmM}{r^2}$$

where m is the mass of the object, M is the mass of the earth, and G is Newton's gravitational constant $G = 6.67 \times 10^{-11} Nm^2/kg^2$.

We can use this relation to calculate an object's escape velocity on the surface of the earth. This is the speed at which an object must be moving away from the earth at the earth's surface if it is to break free from the gravitational attraction of the earth and continue to move away "forever".

Well, we note that if we move away from the earth along a line that goes through the earth's center, then Newton's second law tells us:

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}.$$

By the chain rule we have $\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt}$, and if we note $v = \frac{dr}{dt}$ then we can transform this relation into:

$$v\frac{dv}{dr} = -\frac{GM}{r^2}.$$

If we integrate both sides with respect to r we get:

$$\frac{1}{2}v^2 = \frac{GM}{r} + C.$$

If we set $v(0) = v_0$ and solve this for C we get:

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right).$$

If the object is to escape from the "clutch" of the earth then its velocity must always be positive as $r \to \infty$. This will be the case if

$$v_0 \geq \sqrt{\frac{2GM}{R}}.$$

So, the escape velocity for the earth (or for any planet of a given mass M) is:

$$v_0 = \sqrt{\frac{2GM}{R}}.$$

For the earth the escape velocity is $v_0 \approx 11,180 m/s$.

Example - Suppose that you are stranded - your rocket engine has failed - on an asteroid of diameter 3 miles, with density equal to that of the earth with radius 3960 miles. If you have enough spring in your legs to jump 4 feet straight up on earth while wearing your space suit, can you blast off from this asteroid using leg power alone?

Solution - The escape velocity for the Earth is

$$v_E^2 = \frac{2GM_E}{R_E}.$$

Solving for M_E in this equation we get:

$$M_E = \frac{v_E^2 R_E}{2G}.$$

The density of the earth is its mass divided by its volume:

$$\frac{M_E}{\frac{4}{3}\pi R_E^3} = \frac{3v_E^2}{8\pi G R_E^2}.$$

A similar calculation can be done for the asteroid, and given both the asteroid and the Earth have the same density we get:

$$\frac{3v_E^2}{8\pi G R_E^2} = \frac{3v_A^2}{8\pi G R_A^2}.$$

With a little algebra from this we can deduce the ratio:

$$\frac{v_A}{v_E} = \frac{R_A}{R_E}.$$

So, the escape velocity from the asteroid is:

$$v_A = v_E\left(\frac{R_A}{R_E}\right) = 11,180m/s\left(\frac{1.5miles}{3960miles}\right) = 4.24m/s.$$

On the earth when you jump all your energy is initially kinetic energy, $\frac{1}{2}mv^2$, and at the top of the jump all that energy is converted into potential energy, mgh. So, the initial velocity of the jump is related to the final height by the equation:

$$v = \sqrt{2gh}.$$

Plugging 4 feet in for h we get:

$$v_{Jump} = \sqrt{2(9.8m/s)(4ft)(1m/3.28ft)} = 4.89m/s > 4.24m/s.$$

So, *yes*, you can get off the asteroid! Where you'll go after that, I don't know.

Notes on Homework Problems

Problem 2.3.1 is straightforward.

Problems 2.3.2 and 2.3.4 examine a similar problem, but in 2.3.2 you examine a situation where resistance is proportional to velocity, and in 2.3.4 resistance is proportional to the square of the velocity. What you find is that in the first situation the object stops moving in a finite amount of time, but in the second situation the object keeps moving forever. Keep in mind on these problems that, when $0 < k < 1$ we have $k^2 < k$.

Problem 2.3.10 looks mighty familiar...

Problem 2.3.24 deals with black holes, one of the most amazing things in the universe! Something interesting about black holes is that we can use the classical formula for the escape velocity to derive the radius of the black hole, and the location of what is known as the "event horizon". However, while our equation is correct, the reasoning is totally bogus because a black hole is a very nonclassical situation. In fact, if you analyze a black hole using just special relativity, you find that the classical formula is off by a factor of 2. However, if you then do it correctly using general relativity, an additional factor of 2 is introduced that gives us back our original equation. It's kind of wild that things work out that way, but they do!