Math 2280 - Lecture 8

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Equilibrium Solutions and Stability

Today we're going to talk about the general behavior of *autonomous* differential equations, and how we can extract information about the behavior of these differential equation even when it might be hard or next to impossible to solve them explicitly. Today's lecture corresponds with section 2.2 of the textbook.

The exercises for this section are

Section 2.2 - 1, 10, 21, 23, 24

Introduction

In the previous lecture we examined the simple population growth[1] equation:

$$\frac{dx}{dt} = kx$$

where k is a constant. We also examined the more sophisticated logistic growth equation:

$$\frac{dx}{dt} = kx(M - x)$$

[1]Or, in general, exponential growth.

and saw that these equations were solved, respectively, by the solutions:

$$x(t) = x_0 e^{kx}$$

and

$$x(t) = \frac{Mx_0}{x_0 + (M - x_0)e^{-kMt}}.$$

We were lucky with these equations in that we were able to find explicit solutions without too much bother. Unfortuantely, this isn't always the case. In fact, it's rarely the case. However, even when it's difficult or impossible to solve a differential equation precisely, we can sometimes still get important information about the behavior of the solutions by analyzing the form of the differential equation. Today we're going to talk about ways of doing this for a special type of differential equation called an *autonomous* differential equation.

Autonomous Differential Equations and Phrase Diagrams

A differential equation is called *autonomous* if it has the form:

$$\frac{dx}{dt} = f(x).$$

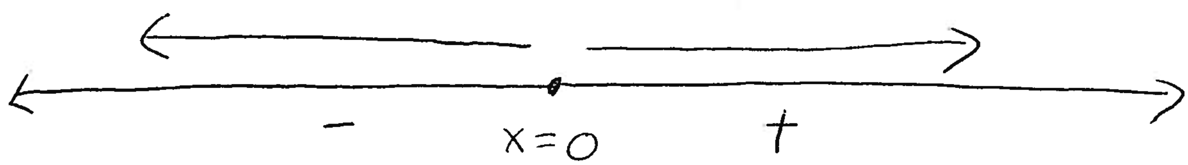
This means that the differential equation does not depend explicitly on the independent variable t, although of course the variable x is a function of t.

Both the population growth equations mentioned above are autonomous differential equations. For each of these we can draw something called a *phase diagram*. These are pictured below for the two differential equations mentioned above.

Differential Equation

$$\frac{dx}{dt} = kx$$

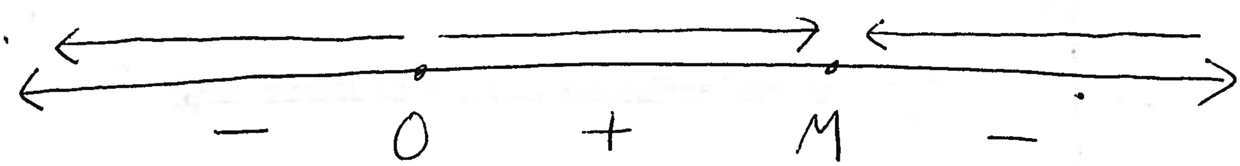
Phase Diagram



Differential Equation

$$\frac{dx}{dt} = kx(M - x)$$

Phase Diagram



Now, what we do to create these phase diagrams is that we solve for the *critical points* of the function $f(x)$. These critical points are the points where the funciton is equal to zero, so the points x such that $f(x) = 0$. In between these critical points, if we assume (as we will) that $f(x)$ is continuous, the function $f(x)$ will be either positive or negative.

To construct a phase diagram we draw a portion of the x-axis containing all the critical points, and we mark the critical points with dots. Then, above the segments and in between these critical points we draw a left arrow if $f(x)$ is negative on the segment, and a right arrow if $f(x)$ is positive on the segment. We also draw the appropriate arrows for the regions greater than any critical point and less than any critical point.

These critical points represent what are called *equilibrium solutions* to our differential equation. These are solutions of the form $x(t) = c$, where c is a constant.

Stability of Critical Points

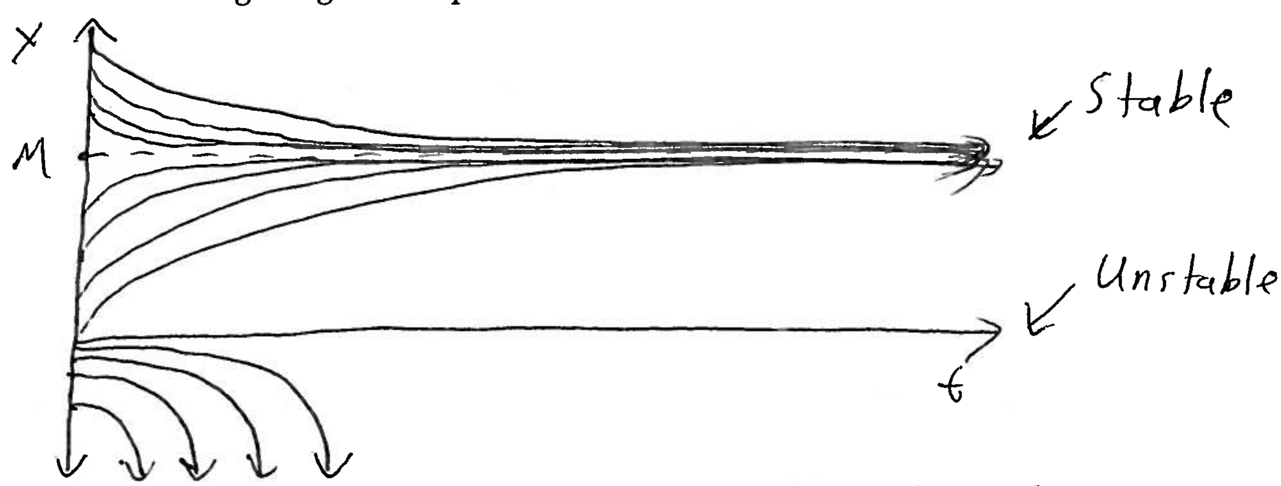
The technical definition of stability of a critical point is this:

Definition - The critical point $x = c$ is *stable* if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$|x_0 - c| < \delta \text{ implies that for all } t > 0 \text{ we have } |x(t) - c| < \epsilon$$

Now, what this is saying is that if you start our sufficiently close to the critical point, within some "band", you'll always stay within that band.

We can see this phenomenon in action if we look at some solution curves for the logistic growth equation:



We can see that for the critical point $x = M$ we have a stable critical point, and that solutions around the point "funnel" towards it. The critical point $x = 0$ on the other hand is an unstable critical point, and we can see that solutions close to it diverge.

Now, it's easy to tell from a phase diagram which critical points are stable and which are not. If your critical point has two arrows going into it, then it's stable. If it has two arrows going away from it, then it's unstable. There can also exist the (rare) situation where a critical point has one arrow going into it and one arrow going out of it. Such a situation we call semistable.

Harvesting a Logistic Population

The autonomouos differential equation:

$$\frac{dx}{dt} = kx(M - x) - h$$

may be considered to describe a logistic population with harvesting. For instance, we might think of the population of fish in a lake from which h fish per year are removed by fishing.

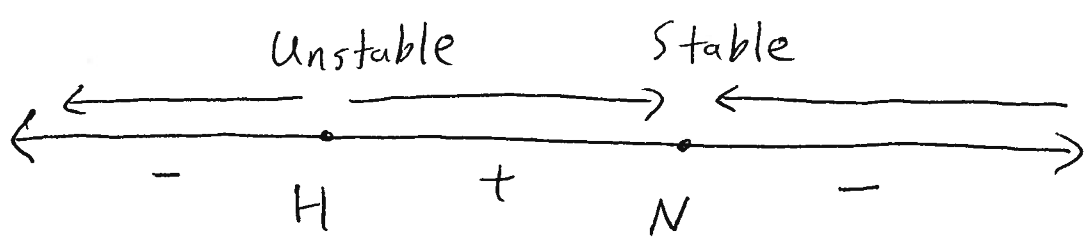
If we solve for the critical points of this differential equation, the quadratic equation tells us these critical points are:

$$c = \frac{kM \pm \sqrt{(kM)^2 - 4hk}}{2k}.$$

If $h < \frac{kM^2}{4}$ then we will have two solutions, call them H and N, where $H < N$. In this case we can rewrite our differential equation as

$$\frac{dx}{dt} = k(N - x)(x - H).$$

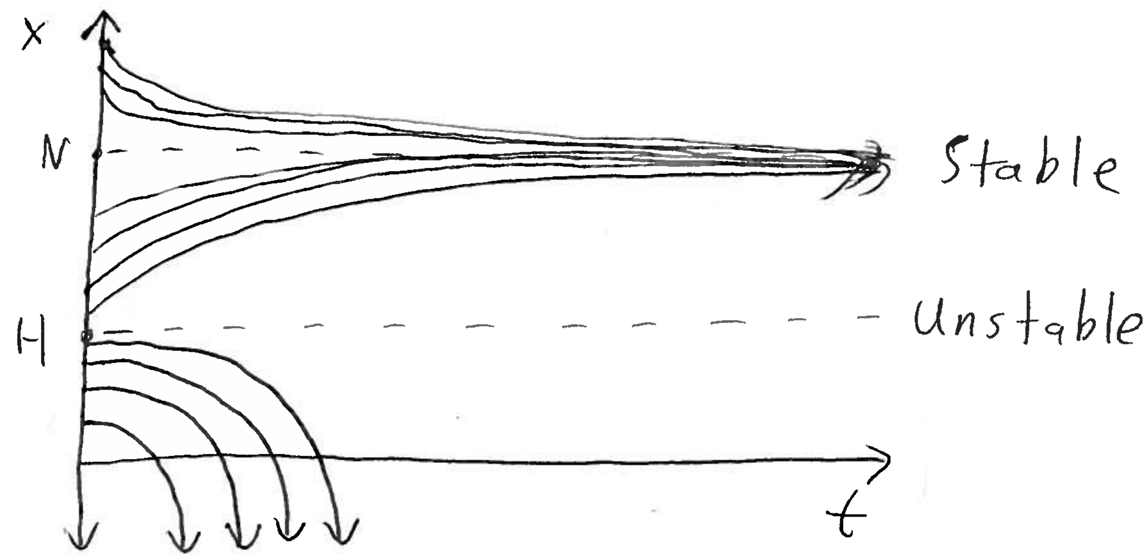
Exercise - Construct the phase diagram for the differential equation above.



The solution to this differential equation (which you'll derive and check as part of your homework) is:

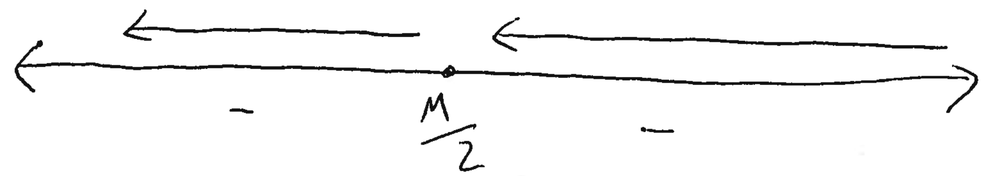
$$x(t) = \frac{N(x_0 - H) - H(x_0 - N)e^{-k(N-H)t}}{(x_0 - H) - (x_0 - N)e^{-k(N-H)t}}.$$

If we graph some representative solution curves of this differential equation we'll get a picture that looks like:

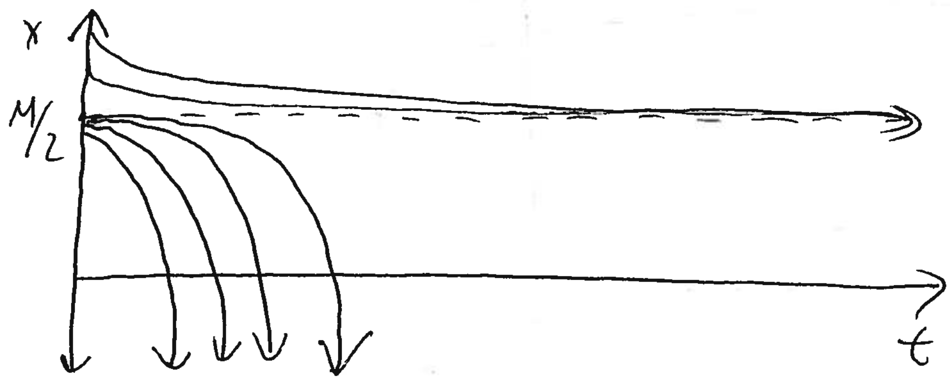


We can see that around N we have a stable critical point, and around H we have an unstable critical point. What this means is that for any initial value greater than H our population size will approach N as time goes on. For any initial value less than H our population size will approach $-\infty$ in a (finite!) amount of time. Of course in reality you can't have less than 0 fish, and so the model would definitely break down when the population becomes sufficiently small.

Now, if $h = \frac{kM^2}{4}$ then we'd have a situation with just one critical point $c = M/2$, and a phase diagram that looks like:

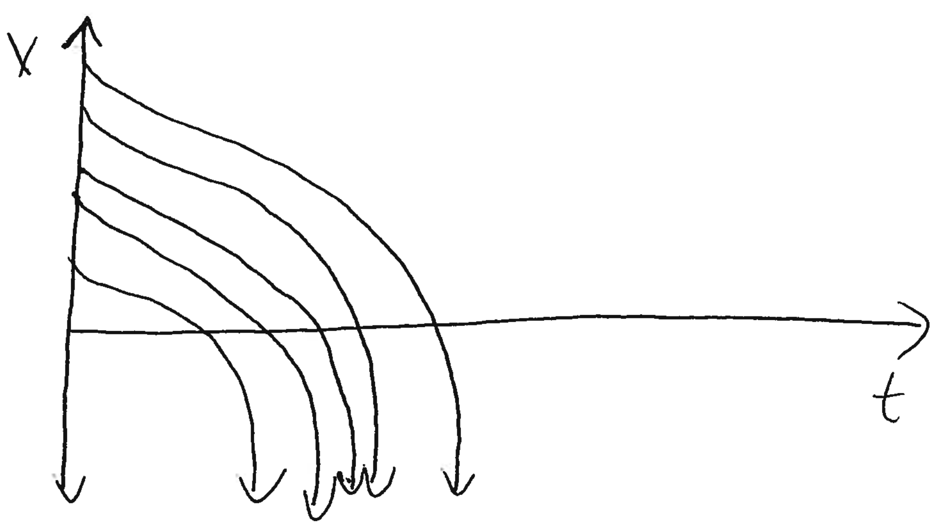


Here our solution curves look like:



and we'd have what's called a semistable equilibrium.

For $h > \frac{kM^2}{4}$ we would have no (real number) critical points, and no matter what our solutions go to $-\infty$ as time increases. These solution curves look like:

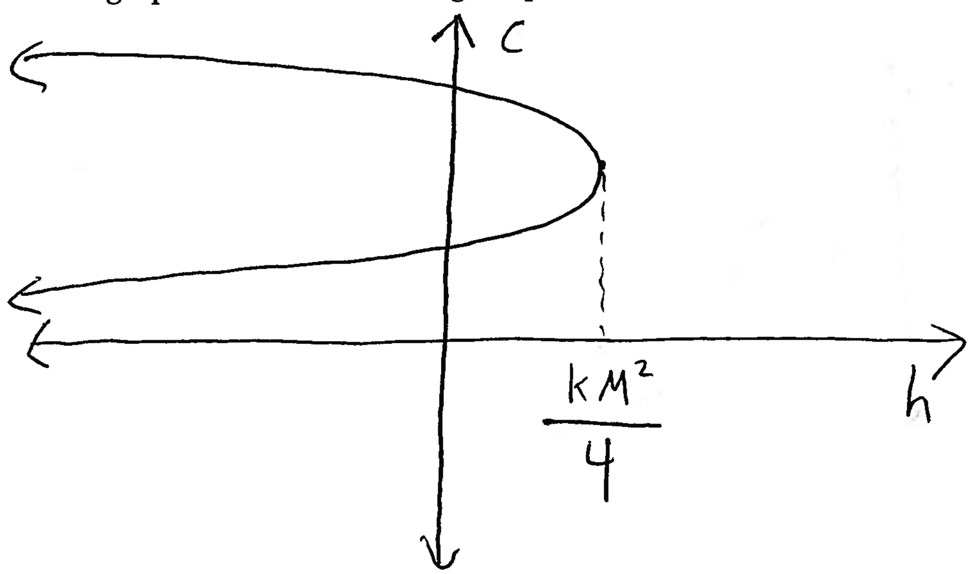


Bifurcation

We can actually see that there's a relation between our critical points and the value of our initial paramater h. The relation can be written as:

$$h = k(Mc - c^2).$$

If we graph this relation we'll get a parabolic curve of the form below:



This is called a bifurcation diagram. It tells us for a given value of h how many critical points we have, and what these critical points will be. These bifurcation diagrams are very important in the study of nonlinear differential equations and chaos.

Notes On Homework Problems

The first two problems from the problem set, problems 2.2.1 and 2.2.10, are straightforward phase diagram problems, and they also ask you to solve relatively simple separable differential equations. Shouldn't be too bad.

Problem 2.2.21 investigates a bifurcation diagram for a different differential equation - the equation $dx/dt = kx - x^3$.

Problem 2.2.23 examines a variation of the harvesting problem from these notes, where instead of fish being removed at a constant rate, the fish are removed at a rate proportional to the fish population. You'll prove that this change actually leaves the differential equation as a logistic differential equation, just with different parameters.

Problem 2.2.24 is where you're actually asked to solve the differential equation for the logistic differential equation with harvesting mentioned above. Hint: It's separable. So, separate the variables, and then use a partial fraction decomposition as we've done in the other problems of this type.