Math 2280 - Lecture 5: Linear First-Order Equations

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Today we're going to examine the first-order version of a type of differential equation that we're going to see quite a bit more of in this class. So, get comfortable with them, because you'll be seeing them again!

This type of differential equation is a *linear* differential equation. A first-order linear differential equation is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Today, we're going to learn how to solve differential equations of this form.

The exercises for this section are:

Section 1.5 - 1, 15, 21, 29, 38, 42

First-Order Linear Differential Equations

When we say a differential equation is *linear*, we mean it's linear in the dependent variable y and its derivatives. So, the equation

$$y' + (e^x \sin x^2)y = x^3 + 2x^2 - 5x + 2$$

is linear, while the differential equation

$$(y')^2 = x$$

is not.

We can multiply both sides of a first-order linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by an *integrating factor*. An integrating factor is a function $\rho(x, y)$ such that, if we multiply both sides by that function, we can recognize both sides of the equation as a derivative. In this case the integrating factor is

$$\rho(x) = e^{\int P(x)dx}.$$

The derivative of ρ is[1]

$$\frac{d\rho}{dx} = P(x)e^{\int P(X)dx}.$$

Using this, we see that the derivative of $ye^{\int P(x)dx}$ is

$$\frac{d}{dx}(ye^{\int P(x)dx}) = e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y.$$

[1]That's the sound of the men working on the chain... rule.

Using this, we see that if we have the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we can multiply both sides by the integrating factor $\rho(x) = e^{\int P(x)dx}$ to get

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x).$$

If we then integrate both sides with respect to x we get

$$e^{\int P(x)dx}y = \int (e^{\int P(x)dx}Q(x))dx + C,$$

which we can then solve for y to get:

$$y(x) = e^{-\int P(x)dx}\left(\int (e^{\int P(x)dx}Q(x))dx + C\right).$$ [2]

Daaaaang! Let's do an example.

[2]The book warns you to *not* memorize this equation. So, whatever you do, don't go memorizing this equation. You should just memorize the method by which we derived the equation. Or, I suppose, in a pinch you could also memorize the equation. But, in practice (at least in this class), things usually aren't as scary as this general solution might make them look.

Example - Solve the initial value problem

$$y' - 2xy = e^{x^2} \qquad\qquad y(0) = 0.$$

Solution - The integrating factor will be:

$$\rho(x) = e^{-\int 2x dx} = e^{-x^2}.$$

Multiplying both sides of the differential equation by ρ we get:

$$e^{-x^2}y' - 2xe^{-x^2}y = 1.$$

Integrating both sides we get:

$$\int \frac{d}{dx}(e^{-x^2}y)dx = \int dx,$$
$$\Rightarrow e^{-x^2}y = x + C.$$

Solving this for $y(x)$ gives us:

$$y(x) = Ce^{x^2} + xe^{x^2}.$$

Plugging in the initial condition $y(0) = 0$ we get:

$$y(0) = Ce^{0^2} + 0e^{0^2} = C = 0.$$

So, the solution to the initial value problem is:

$$y(x) = xe^{x^2}.$$

Existence, Uniqueness, and Examples

Now, again, before we spend too long trying to solve a differential equation, we'd like to know whether or not a solution even exists, and if it does exist, if the solution is unique. For linear differential equations, we have a theorem that's even nicer than our result from section 1.3.

Theorem - If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0$$

has a unique solution $y(x)$ on I.

Note that we're guaranteed a unique solution on the *entire* interval I, not just on some possibly smaller interval like we had for the theorem from section 1.3. Linear differential equations are nice that way.

As a first application of linear first-order equations, we consider a tank containing a solution - a mixture of solute and solvent - such as salt dissolved in water. There is both inflow and outflow, and we want to compute the *amount* $x(t)$ of solute in the tank at time t, given the amount $x(0) = x_0$ at time $t = 0$. Suppose that solution with a concentration of c_i grams of solute per liter of solution flows into the tank at the constant rate of r_i liters per second, and that the solution in the tank - kept thoroughly mixed by stirring - flows out at the constant rate of r_o liters per second.

The amount of solute flowing into the tank will be

$$r_i c_i,$$

while if c_o is the concentration of the outgoing solution the amount of solute flowing out of the tank will be

$$r_o c_o.$$

So, if $x(t)$ represents the amount of solute in the tank, its rate of change will be:

$$\frac{dx}{dt} = r_i c_i - r_o c_o.$$

Now, we'll usually assume r_i, r_o, and c_i are constant, but the output concentration might very well be changing over time. So, c_o will be given by

$$c_o = \frac{x(t)}{V(t)}.$$

Here $V(t)$ is the volume of water in the tank, which itself might be changing over time. Well, if we plug this in for c_o we get a linear first-order differential equation! Namely,

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V} x.$$

Example - A 120-gallon (gal) tank initially contains 90 lbs of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at a rate of 4 gal/min, and the well-stirred mixture flows out of the tank at a rate of 3 gal/min. How much salt does the tank contain when it is full?

Solution - The differential equation modeling this system, where $x(t)$ is the amount of salt in the solution at time t, is:

$$\frac{dx}{dt} = (4\ gal/min)(2lb/gal) - \frac{(3\ gal/min)}{90+t}x(t).$$

We can rewrite this as:

$$\frac{dx}{dt} + \frac{3}{90+t}x = 8.$$

The integrating factor here will be:

$$\rho = e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = (90+t)^3.$$

Multiplying both sides by ρ and integrating we get:

$$\int \frac{d}{dt}((90+t)^3 x)dt = \int 8(90+t)^3 dt$$
$$\Rightarrow (90+t)^3 x(t) = 2(90+t)^4 + C.$$

So, the function $x(t)$ will be:

$$x(t) = 2(90+t) + \frac{C}{(90+t)^3}.$$

If we plug in our initial condition we get:

$$x(0) = 2(90+0) + \frac{C}{90^3} = 90.$$
$$\Rightarrow C = -(90^4).$$

So, our solution is:

$$x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

If we plug in $t = 30$ we get:

$$x(30) = 2(90+30) - \frac{90^4}{(90+30)^3} \approx 202 \text{ lbs of salt.}$$

Notes on Homework Problems

The first three assigned problems, 1.5.1, 1.5.15, and 1.5.21 are there so you can get some practice using the integrating factor method. Shouldn't be too bad.

The fourth problem, 1.5.29, is a pretty cool problem that involves the "error function", a function that is absoutely ubiquitous in probability and statistics given its relation to the central limit theorem. It's also not a function that can be expressed in terms of the standard functions (exponentials, logarithms, polynomials, sines, cosines, etc...) that you've learned to know and love. It's a "special function", of which you'll see many if you take higher level engineering, physics, or applied math classes.

Problem 1.5.38 is a bit involved, but not terribly difficult if you set it up correctly. Take some time logically thinking through how to set it up! Once it's begun correctly, it's not so bad. Also, the third part of the problem tests your optimization skillz from calculus I.

Problem 1.5.42 is a nifty little problem with a pretty simple end result. I thought it was fun!