Math 2280 - Lecture 4: Separable Equations and Applications

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For the last two lectures we've studied first-order differential equations in standard form:

$$y' = f(x, y).$$

We learned how to solve these differential equations for the special situation where $f(x, y)$ is independent of the variable y, and is just a function of x, so $f(x, y) = f(x)$. We also learned about slope fields, which give us a geometric method for understanding solutions and approximating them, even if we cannot find them directly.

Today we're going to discuss how to solve first-order differential equations in standard form in the special situation where the function $f(x, y)$ is *separable*, which means we can write $f(x, y)$ as the product of a funciton of x, and a function of y.

The exercises for this section are:

Section 1.4 - 1, 3, 17, 19, 31, 35, 53, 68

Separable Equations and How to Solve Them

Suppose we have a first-order differential equation in standard form:

$$\frac{dy}{dx} = h(x, y).$$

If the function $h(x, y)$ is *separable* we can write it as the product of two functions, one a function of x, and the other a function of y. So,

$$h(x, y) = \frac{g(x)}{f(y)}.$$

In this situation we can manipulate our differtial equation to put everything with a y term on one side, and everything with an x term on the other:

$$f(y)dy = g(x)dx.$$

From here we can just integrate both sides of the equation, and then solve for y as a funciton of x. Easy!

For example, suppose we're given the differential equation

$$\frac{dP}{dt} = P^2.$$

We can rewrite this equation as

$$\frac{dP}{P^2} = dt,$$

and then integrate both sides of the equation to get

$$-\frac{1}{P} = t + C.$$

Solving this for P as a function of t gives us

$$P(t) = \frac{1}{C - t}.^{1}$$

Note that this function has a vertical asymptote as t approaches C. If this is a population model, this is called *doomsday*!

Examples of Separable Differential Equations

Suppose we're given the differential equation

$$\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}.$$

This differential equation is separable, and we can rewrite it as

$$(3y^2 - 5)dy = (4 - 2x)dx.$$

If we integrate both sides of this differential equation

$$\int (3y^2 - 5)dy = \int (4 - 2x)dx$$

we get

$$y^3 - 5y = 4x - x^2 + C.$$

This *is* a solution to our differential equation, but we cannot readily solve this equation for y in terms of x. So, our solution to this differential equation must be implicit.

[1] Note that we're playing a little fast and loose with the unknown constant C here. In particular, if we multiply an unknown constant C by -1, it's still just an unknown constant, and we continue to call it (positive) C.

If we're given an initial value, say $y(1) = 3$, then we can easily solve for the unknown constant C:

$$3^3 - 5(3) = 4(1) - 1^2 + C \Rightarrow C = 9.$$

So, around the point $(1, 3)$ the differential equation will have the unique solution given implicitly by the curve defined by

$$y^3 - 5y = 4x - x^2 + 9.$$

Example - Solve the differential equation

$$\frac{dy}{dx} = 6x(y-1)^{\frac{2}{3}}.$$

Solution - This is a separable differential equation, and we can rewrite it as:

$$\frac{dy}{(y-1)^{\frac{2}{3}}} = 6x dx.$$

Integrating both sides we get:

$$\int \frac{dy}{(y-1)^{\frac{2}{3}}} = \int 6x dx$$
$$\Rightarrow 3(y-1)^{\frac{1}{3}} = 3x^2 + C.$$

Solving this for y we get:

$$y(x) = (x^2 + C)^3 + 1.$$

Now, we note that $y(x) = 1$ is *also* a solution to this differential equation. So, if we're given the initial condition $y(0) = 1$ we have two solutions, namely:

$$y_1(x) = x^6 + 1 \quad \text{and} \quad y_2(x) = 1.$$

So, what gives? Well, the reason we can have two solutions is that while the function

$$f(x, y) = 6x(y-1)^{\frac{2}{3}}$$

is continuous everywhere, its partial derivative

$$\frac{\partial f}{\partial y} = \frac{4x}{(y-1)^{\frac{1}{3}}}$$

is undefined where $y = 1$. So, for any initial condition $y(a) = b$ where $b \neq 1$ there is, locally, a unique solution. But, for $b = 1$, there is not.

A very common, and simple, type of differential equation that is used to model many, many things[2] is:

$$\frac{dx}{dt} = kx,$$

where k is some constant.

Now, this is a separable equation, and so it can be solved by our methods. First, we rewrite it as

$$\frac{dx}{x} = kdt,$$

and then integrate both sides

$$\int \frac{dx}{x} = \int kdt$$

[2]Compound interest, population growth, radioactive decay, etc...

to get

$$\ln x = kt + C.$$

If we then exponentiate both sides we get

$$x(t) = e^{kt+C} = e^C e^{kt} = Ce^{kt}.^3$$

So, the solution to our differential equation is exponential growth (if $k > 0$) or exponential decay (if $k < 0$). If $k = 0$ the answer is just a boring unknown constant.

Radioactive decay is quite accurately measured by an exponential decay function. For ^{14}C decay, the decay constant is $k \approx -0.0001216$ if t is measured in years.

Example - Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present-day specimen of the same substance contained 5.0×10^{10} atoms of ^{14}C per gram. Compute the approximate age of the relic. What is your opinion as to its authenticity?

Solution - The equation for the amount of ^{14}C will be:

$$C(t) = C_0 e^{kt},$$

where $C_0 = 5.0 \times 10^{10}$ atoms, and $k = -0.0001216$. We're given that

$$C(t_0) = 4.6 \times 10^{10} \text{ atoms.}$$

So, this means:

[3] The American Society for the Prevention of Notation Abuse would strongly protest this last equality. I'm just saying that e^C, where C is an unknown constant, is itself just an unknown constant, and I don't like having to come up with new letters, so I just continue to represent the unknown constant as C.

$$4.6 \times 10^{10} = (5.0 \times 10^{10})e^{-.0001216 t_0}.$$

Solving this for t_0 we get:

$$t_0 = \frac{\ln\left(\frac{4.6\times 10^{1-}}{5.0\times 10^{10}}\right)}{-0.0001216} \approx 685.7 \text{ years.}$$

So, based upon this test, if our assumptions are correct it would appear the relic is not from the time of Christ. Still pretty old, but not nearly that old.

Notes on Homework Problems

Problems 1.4.1, 1.4.3, 1.4.17, and 1.4.19 are all straightforward separable differential equations like the examples above.

Problem 1.4.31 investigates the subtle distinctions between two seeminly very similar differential equations.

Problem 1.4.35 and 1.4.53 are standard radiocarbon dating problems. Shouldn't be too hard.

Problem 1.4.68 is a challenge! It's an introduction to one of the most awesome problems in the history of mathematics, the *brachistochrone*! It's also the problem that led to a field of analysis called the "calculus of variations", which is extremely important in physics. In fact, it's one of my favorite ideas in all of nature! There's a lecture in volume 1 of The Feynman Lectures on Physics about the principle of least action that I'd strongly recommend you read. This problem won't be graded, but I hope you give it some effort.