Math 2280 - Lecture 46

Dylan Zwick

Fall 2013

Today we're going to use the tools we've developed in the last two lectures to analyze some systems of nonlinear differential equations that arise in simple ecological models.

The idea behind these models is that there are two different species living within and interacting within the same environment. Now, our models will be very simple; far, far simpler than any real ecological system. However, the models still exhibit some interesting qualitative behavior that can help us understand what happens in actual ecological systems. Plus, the models are interesting in their own right as differential equations.

The assigned problems for this section, and the last assigned problems for this course, are:

$$\text{Section 6.3 - 3, 4, 5, 6, 7}$$

The Predator-Prey System

Suppose we have two species, species x and species y, and these two species interact within an ecological system. For this system, we'll assume that species x is an organism that feeds upon some other organism abundant in the environment, while species y is an organism that feeds upon, well, special x! So, species y is a predator, and species x is the prey. We'll make the following assumptions.

- Without predators, the prey population would grow at a rate directly proportional to its population $dx/dt = ax$.

- Without prey, the predator population would decrease at a rate directly proportional to its population $dy/dt - -by$.

- When both predators and prey are present, in combination with these natural rates of growth and decline, the size of the predator population increases with the size of the prey population, and the size of the prey population decreases with the size of the predator population.

Stated mathematically, we have the system of differential equations:

$$\frac{dx}{dt} = ax - pxy = x(a - py),$$

$$\frac{dy}{dt} = -by + qxy = y(-b + qx).$$

This is known as a *predator-prey system*.

This system has two critical points, $(0, 0)$, and $(b/q, a/p)$.

Our Jacobian matrix for this system is:

$$J(x, y) = \begin{pmatrix} a - py & -px \\ qy & -b + qx \end{pmatrix}.$$

At $(0, 0)$ this Jacobian is:

$$J(0, 0) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}.$$

This matrix has eigenvalues $\lambda_1 = a > 0$, and $\lambda_2 = b < 0$. So, the point $(0, 0)$ is an unstable saddle point.

The critical point $(b/q, a/p)$ gives us the Jacobian matrix:

$$J(b/q, a/p) = \begin{pmatrix} 0 & -\frac{pb}{q} \\ \frac{aq}{p} & 0 \end{pmatrix}.$$

This Jacobian has the pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm i\sqrt{ab}$. So, the linearlization is a stable center, and the actual behavior is undetermined, as it could be either a stable center, an unstable spiral sink, or a stable spiral source. If we draw the phase portrait for this system it looks like:

Note:
Check the
textbook for
a **much** better
drawing.
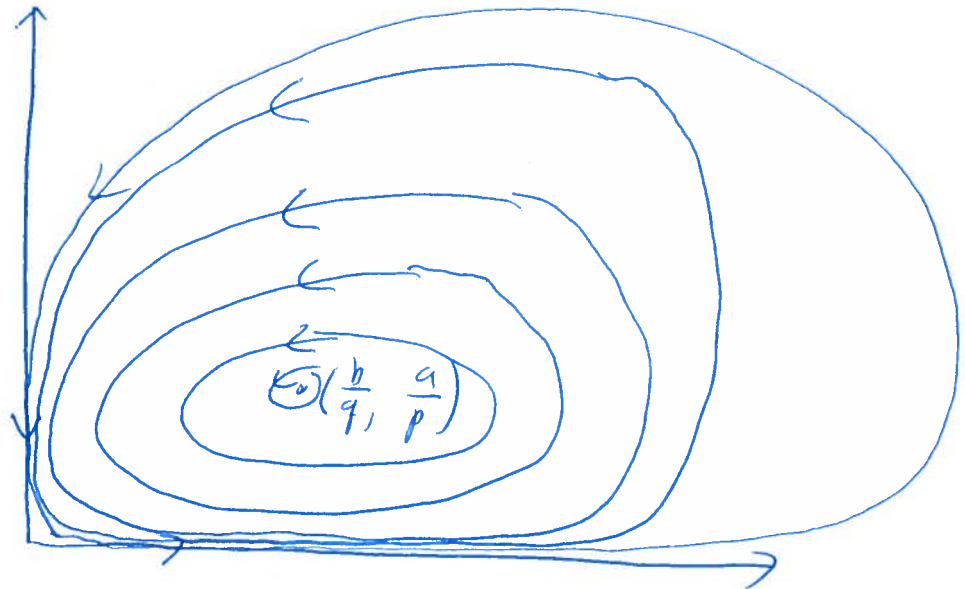
So, it appears the point $(b/q, a/p)$ is, in fact, a stable center.

Competing Species

Let's now consider a system in which we have two populations x and y, and on their own both populations satisfy the logistic growth equations:

$$\frac{dx}{dt} = a_1 x - b_1 x^2,$$

$$\frac{dy}{dt} = a_2 y - b_2 y^2.$$

When these populations share an environment they compete, and each negatively impacts the growth of the other, creating the interacting system of differential equations:

$$\frac{dx}{dt} = a_1 x - b_1 x^2 - c_1 xy = x(a_1 - b_2 x - c_1 y),$$

$$\frac{dy}{dt} = a_2 y - b_2 y^2 - c_2 xy = y(a_2 - b_2 y - c_2 x).$$

This is a *competition system*, and it has four critical points. Three of these points are: $(0,0); (0, a_2/b_2); (a_1/b_1, 0)$. The fourth critical point will be the simultaneous solution to the system:

$$\begin{array}{ccccc} b_1 x & + & c_1 y & = & a_1 \\ c_2 x & + & b_2 y & = & a_2 \end{array}.$$

We'll call this fourth point (x_E, y_E). We'll assume there is a unique solution (so the two equations are not linearly dependent) and the point (x_E, y_E) is in the first quadrant. We want to know if the critical point (x_E, y_E) is stable, and it turns out this depends upon the relative values of $c_1 c_2$ and $b_1 b_2$. If $c_1 c_2 < b_1 b_2$, it's a stable critical point, and all initial populations where both species are present will approach it. If $b_1 b_2 < c_1 c_2$ then it's an unstable critical point, and any initial population with both species present will exhibit extinction of one or the other species.

Example - Find the critical point (x_E, y_E) for the system:

$$\frac{dx}{dt} = 14x - \frac{1}{2}x^2 - xy,$$

$$\frac{dy}{dt} = 16y - \frac{1}{2}y^2 - xy.$$

Will the critical point be stable or unstable? Draw the corresponding phase portrait for this system.

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More room for the example problem.

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Will the critical point be stable or unstable? Draw the corresponding phase portrait for this system.

Notes on Homework Problems

Problem 6.3.3 is very interesting, and shows how it's possible that, by attempting to decrease the population of a harmful species in a population, it's possible to do just the opposite!

Problems 6.3.4 through 6.3.7 step you through an analysis of all four critical points of a competition system. These problems can all be viewed as part of one big problem. Taken as individual parts, it shouldn't be too bad, and is very similar to the example problems from the textbook.