Math 2280 - Lecture 45

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Last time, in lecture 44, we introduced two-dimensional autonomous systems of differential equations. Unlike almost all the differential equations we've studied so far in this class, these differential equations were not assumed to be linear. We looked at some phase diagrams, defined critical points, and classified the types of critical points.

Today, we're going to take a closer look at how we analyze the behavior of a system around a critical point, and how we can mathematically determine the type of critical point.

Today's lecture corresponds with section 6.2 from the textbook, and the assigned problems for this section are:

Section 6.2 - 1, 5, 8, 15, 31

Linearization

We'll assume throughout that any critical point we're examining is an *isolated critical point*, which means in a small enough region around the critical point, there are no other critical points.

We'll also frequently assume that the isolated critical point is the point $(0,0)$. This isn't as restrictive as you might think[1] in that for any other critical point (a, b) we can just shift our system with the transformation $u = x - a$, and $v = y - b$. The shifted system will have the same behavior

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[1] In fact, it's not restrictive at all.

as the original system, just with all x-values shifted to the left by a, and all y-values shifted down by b.

If our functions $f(x, y)$ and $g(x, y)$ are continuously differentiable at the critical point (x_0, y_0) then around this point we can approximate the functions by:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)x + f_y(x_0, y_0)y,$$
$$g(x, y) \approx g(x_0, y_0) + g_x(x_0, y_0)x + y_y(x_0, y_0)y.$$

As $f(x_0, y_0) = g(x_0, y_0) = 0$ at a critical point by definition, around a critical point we have:

$$\frac{dx}{dt} = f_x(x_0, y_0)x + f_y(x_0, y_0)y,$$
$$\frac{dy}{dt} = g_x(x_0, y_0)x + g_y(x_0, y_0)y.$$

We can write this in matrix notation as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{J}(x_0, y_0) \begin{pmatrix} x \\ y \end{pmatrix}.$$

Here $\mathbf{J}(x_0, y_0)$ is the *Jacobian* matrix you may remember from multivariable calculus.[2]

The behavior of our system around a critical point depends on this Jacobian matrix. In particular, it depends upon the eigenvalues of the Jacobian matrix.

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[2]Or, you may not remember it.

Critical Points and Eigenvalues

To determine the behavior of our system around a critical point, we look at the eigenvalues of the Jacobian matrix at that critical point. We will assume our critical point is $(0,0)$, and that the critical point is isolated, which means $\lambda = 0$ will not be an eigenvalue of the Jacobian matrix.[3]

The five possibilities for the two eigenvalues are are:

- real and unequal with the same sign;

- real and unequal with opposite sign;

- real and equal;

- complex conjugates with nonzero real part;

- pure imaginary numbers.

We'll examine each of these in turn, in the context of an example.

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[3]Why?

Example - Suppose $(0, 0)$ is a critical point of our system, and the Jacobian matrix at $(0, 0)$ is:
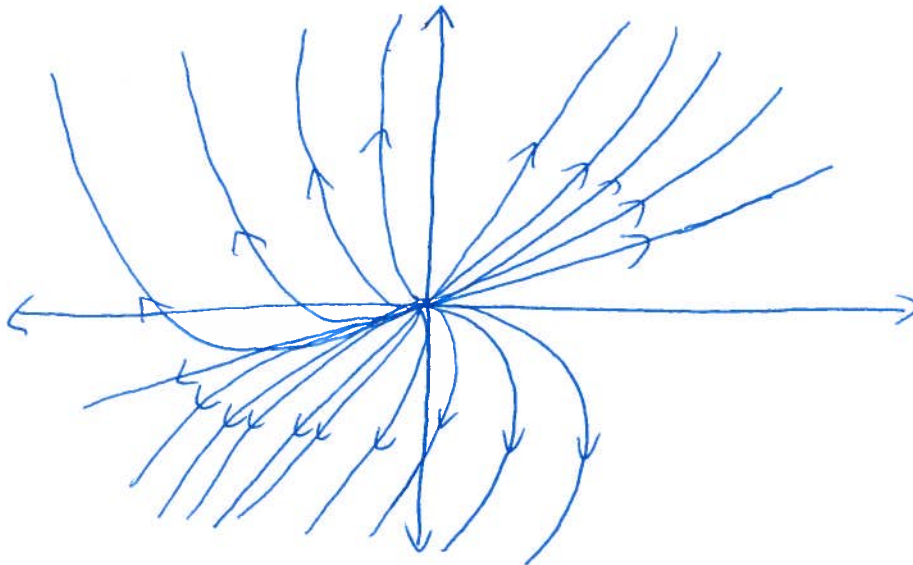
$$\frac{1}{8} \begin{pmatrix} 7 & 3 \\ -3 & 17 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Solution - The eigenvalues of this matrix are $\lambda = 1$ and $\lambda = 2$, with associated eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The correspoding phase diagram looks like:

From here we can see that $(0, 0)$ is an *improper nodal source*.

In general, if λ_1, λ_2 are both positive, then we have an improper nodal source, and if they're both negative, we have an improper nodal sink.

Example - Suppose $(0,0)$ is a critical point of our system, and the Jacobian matrix at $(0,0)$ is:
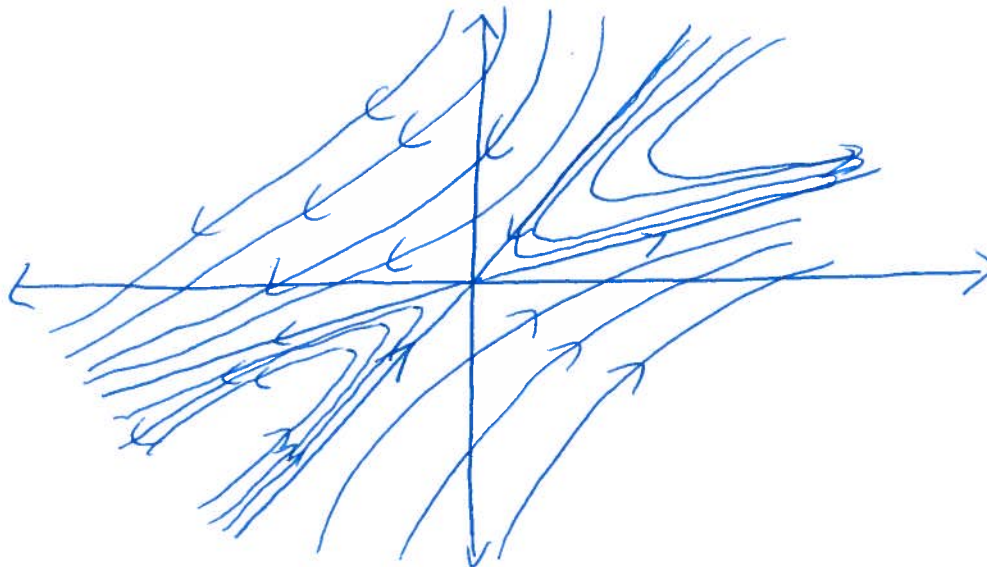
$$\frac{1}{4}\begin{pmatrix} 5 & -3 \\ 3 & -5 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Solution - Here the eiganvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, with associated eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The correspoding phase diagram looks like:

From here we can see that $(0,0)$ is a *saddle point*.

In general, if λ_1 and λ_2 are real with opposite sign then the critical point is a saddle point.

Example - Suppose $(0,0)$ is a critical point of our system, and the Jacobian matrix at $(0,0)$ is:

$$\frac{1}{8} \begin{pmatrix} -11 & 9 \\ -1 & -5 \end{pmatrix}.$$

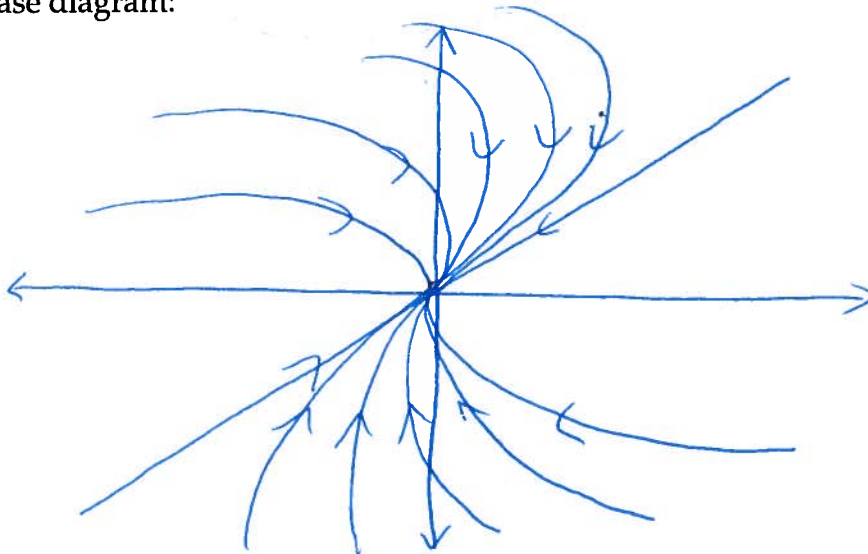What is the behavior of our solution curves around this critical point?

Solution - Here the eiganvalues are $\lambda_1 = -1$ and $\lambda_2 = -1$, with associated eigenvector:

$$\mathbf{v_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The vector

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

is a generalized eigenvector based on $\mathbf{v_1}$, but only $\mathbf{v_1}$ shows up in the phase diagram:

From here we can see that $(0,0)$ is an *improper nodal sink.*

6

In general, if λ_1 and λ_2 are real and equal, then if there are two linearly independent eigenvectors the critical point is a proper node, and if not the critical point is an improper node. If λ_1, λ_2 are positive the critical point is a source, while if they're negative the critical point is a sink.

Example - Suppose $(0,0)$ is a critical point of our system, and the Jacobian matrix at $(0,0)$ is:

$$\frac{1}{4} \begin{pmatrix} -10 & 15 \\ -15 & 8 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Solution - Here the eiganvalues are $\lambda = -\frac{1}{4} \pm 3i$. The corresponding phase diagram looks like:
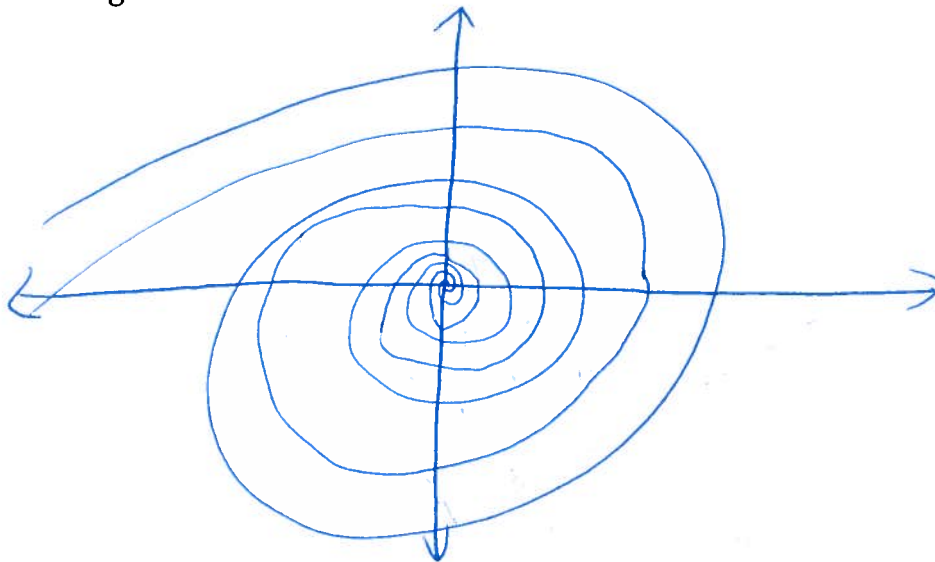
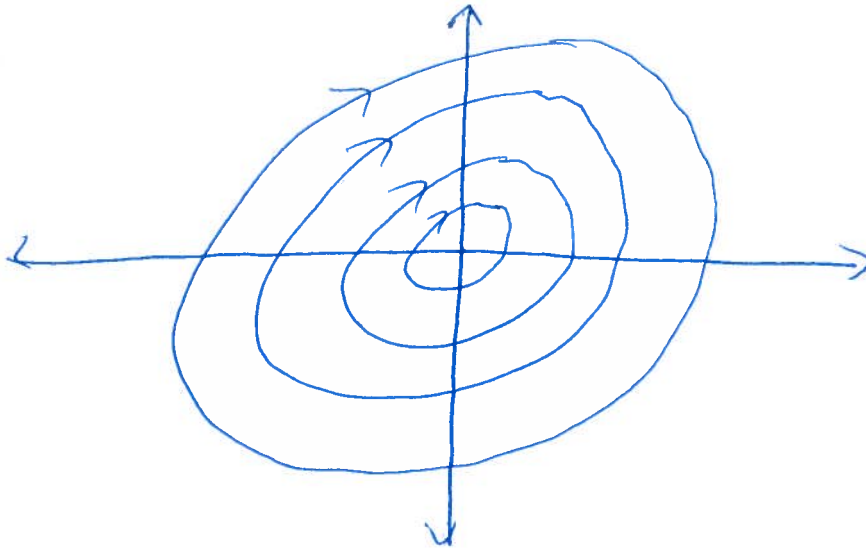From here we can see that $(0,0)$ is a *saddle point*, and a sink.

In general, if λ_1 and λ_2 are complex conjugates then the critical point is a spiral point. If the real part is negative it's a sink, and if the real part is positive it's a source.

Example - Suppose $(0,0)$ is a critical point of our system, and the Jacobian matrix at $(0,0)$ is:

$$\frac{1}{4}\begin{pmatrix} -9 & 15 \\ -15 & 9 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Solution - Here the eiganvalues are $\lambda = \pm 3i$. The correspoding phase diagram looks like:

From here we can see that $(0,0)$ is a *stable center*.

Stability

The eigenvalues of the Jacobian not only tell us the type of critical point, but these eigenvalues also tell us whether or not the critical point is stable. If λ_1, λ_2 are the eigenvalues of the Jacobian at a critical point then:

1. The critical point is asymptotically stable if the real parts of both eigenvalues are negative.

2. The critical point is stable but not asymptotically stable if the real parts are both zero. (So the eigenvalues are pure imaginary.)

3. The critical point is unstable if either eigenvalues has a positive real part.

We can see examples of all thes possibilities in those examined above.

Notes on Homework Problems

Exercises 6.2.1, 6.2.5, 6.2.8, 6.2.15, and 6.2.31 are all about finding and clasifying critical points. Shouldn't be too hard. Don't worry about verifying your conclusion with a computer system or graphing calculator.