

Math 2280 - Lecture 45

Dylan Zwick

Fall 2013

Last time, in lecture 44, we introduced two-dimensional autonomous systems of differential equations. Unlike almost all the differential equations we've studied so far in this class, these differential equations were not assumed to be linear. We looked at some phase diagrams, defined critical points, and classified the types of critical points.

Today, we're going to take a closer look at how we analyze the behavior of a system around a critical point, and how we can mathematically determine the type of critical point.

Today's lecture corresponds with section 6.2 from the textbook, and the assigned problems for this section are:

Section 6.2 - 1, 5, 8, 15, 31

Linearization

We'll assume throughout that any critical point we're examining is an *isolated critical point*, which means in a small enough region around the critical point, there are no other critical points.

We'll also frequently assume that the isolated critical point is the point $(0, 0)$. This isn't as restrictive as you might think¹ in that for any other critical point (a, b) we can just shift our system with the transformation $u = x - a$, and $v = y - b$. The shifted system will have the same behavior

¹In fact, it's not restrictive at all.

as the original system, just with all x -values shifted to the left by a , and all y -values shifted down by b .

If our functions $f(x, y)$ and $g(x, y)$ are continuously differentiable at the critical point (x_0, y_0) then around this point we can approximate the functions by:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)x + f_y(x_0, y_0)y,$$

$$g(x, y) \approx g(x_0, y_0) + g_x(x_0, y_0)x + g_y(x_0, y_0)y.$$

As $f(x_0, y_0) = g(x_0, y_0) = 0$ at a critical point by definition, around a critical point we have:

$$\frac{dx}{dt} = f_x(x_0, y_0)x + f_y(x_0, y_0)y,$$

$$\frac{dy}{dt} = g_x(x_0, y_0)x + g_y(x_0, y_0)y.$$

We can write this in matrix notation as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{J}(x_0, y_0) \begin{pmatrix} x \\ y \end{pmatrix}.$$

Here $\mathbf{J}(x_0, y_0)$ is the *Jacobian* matrix you may remember from multivariable calculus.²

The behavior of our system around a critical point depends on this Jacobian matrix. In particular, it depends upon the eigenvalues of the Jacobian matrix.

²Or, you may not remember it.

Critical Points and Eigenvalues

To determine the behavior of our system around a critical point, we look at the eigenvalues of the Jacobian matrix at that critical point. We will assume our critical point is $(0, 0)$, and that the critical point is isolated, which means $\lambda = 0$ will not be an eigenvalue of the Jacobian matrix.³

The five possibilities for the two eigenvalues are are:

- real and unequal with the same sign;
- real and unequal with opposite sign;
- real and equal;
- complex conjugates with nonzero real part;
- pure imaginary numbers.

We'll examine each of these in turn, in the context of an example.

³Why?

Example - Suppose $(0, 0)$ is a critical point of our system, and the Jacobian matrix at $(0, 0)$ is:

$$\frac{1}{8} \begin{pmatrix} 7 & 3 \\ -3 & 17 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Example - Suppose $(0, 0)$ is a critical point of our system, and the Jacobian matrix at $(0, 0)$ is:

$$\frac{1}{4} \begin{pmatrix} 5 & -3 \\ 3 & -5 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Example - Suppose $(0, 0)$ is a critical point of our system, and the Jacobian matrix at $(0, 0)$ is:

$$\frac{1}{8} \begin{pmatrix} -11 & 9 \\ -1 & -5 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Example - Suppose $(0, 0)$ is a critical point of our system, and the Jacobian matrix at $(0, 0)$ is:

$$\frac{1}{4} \begin{pmatrix} -10 & 15 \\ -15 & 8 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Example - Suppose $(0, 0)$ is a critical point of our system, and the Jacobian matrix at $(0, 0)$ is:

$$\frac{1}{4} \begin{pmatrix} -9 & 15 \\ -15 & 9 \end{pmatrix}.$$

What is the behavior of our solution curves around this critical point?

Stability

The eigenvalues of the Jacobian not only tell us the type of critical point, but these eigenvalues also tell us whether or not the critical point is stable. If λ_1, λ_2 are the eigenvalues of the Jacobian at a critical point then:

1. The critical point is asymptotically stable if the real parts of both eigenvalues are negative.
2. The critical point is stable but not asymptotically stable if the real parts are both zero. (So the eigenvalues are pure imaginary.)
3. The critical point is unstable if either eigenvalues has a positive real part.

We can see examples of all these possibilities in those examined above.

Notes on Homework Problems

Exercises 6.2.1, 6.2.5, 6.2.8, 6.2.15, and 6.2.31 are all about finding and classifying critical points. Shouldn't be too hard. Don't worry about verifying your conclusion with a computer system or graphing calculator.