Math 2280 - Lecture 44

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In this class we focus almost exclusively on linear differential equations. There are many reasons for this: linear differential equations are quite interesting; many physical systems can be approximated using linear differential equations; linear differential equations are A LOT easier to solve exactly than nonlinear differential equations. However, it's only slightly facetious to say that differential equations divide into linear and nonlinear types in the same way that objects in the universe divide into bananas and not-bananas!

So, I'd be negligent if I didn't at least talk a little bit about nonlinear differential equations at some point in this course. I'll spend our last three or four lectures doing so. You won't be required to know this material for the exam, although there might be an extra credit problem related to it. There will be a homework assignment, but it will also be for extra credit.

Today's lecture corresponds with section 6.1 from the textbook. The assigned problems for this section are:

Section 6.1 - 1, 5, 10, 18, 30

Two-Dimensional Autonomous Systems

A wide variety of natural phenomenon are modeled by a two-dimensional first-order system of the form

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

in which the independent variable t does not explicitly appear. Such systems, in which t does not appear, are called *autonomous* systems. We will generally assume that the functions F and G are continuously differentiable within some region R of the xy-plane. If this is the case then given any initial time t_0 and any initial point (x_0, y_0) within R, there is a unique solution $x = x(t)$, $y = y(t)$ that is defined on some open interval of "time" containing t_0. These equations describe a parametrized solution curve in the phase plane.

A *critical point* of the system is a point (x_*, y_*) such that:

$$F(x_*, y_*) = G(x_*, y_*) = 0.$$

At a critical point a constant solution satisfies the differential equation. Namely, the solution that begins and then just stays at the critical point. This is called an *equilibrium solution*.

Example - Find the critical points of the autonomous system:

$$\frac{dx}{dt} = x - 2y + 3$$
$$\frac{dy}{dt} = x - y + 2$$

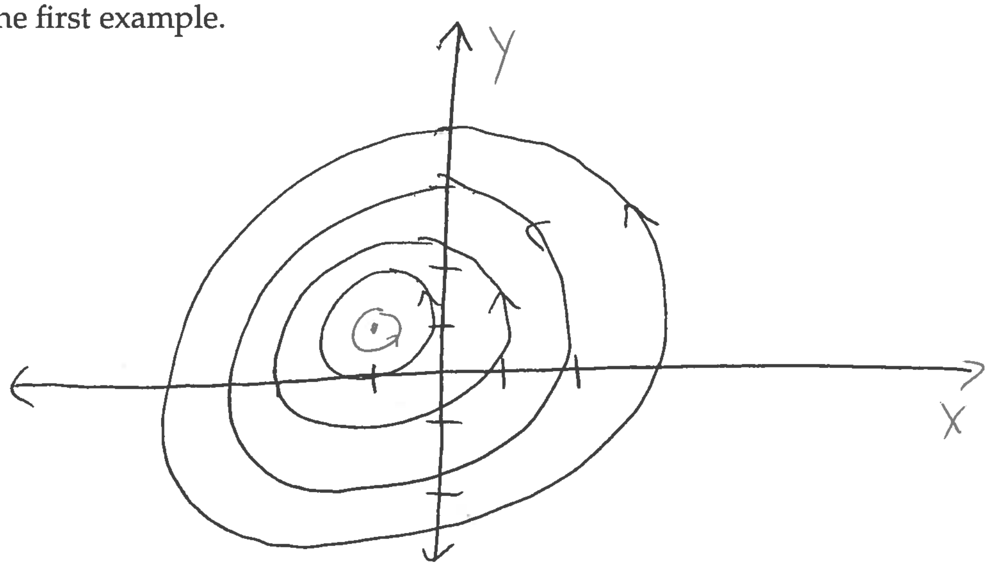
Solution - There will only be one critical point of this system, and that will be at $(-1, 1)$.

Phase Portraits

If the initial point (x_0, y_0) is not a critical point, then the corresponding trajectory is a curve in the xy-plane. It turns out that such a curve will be a nondegenerate curve with no self intersections. A picture that shows an autonomous system's stable points, along with a collection of typical solution curves (trajectories) is called a *phase portrait*. We can also visualize this system by constructing a *slope field* in the xy-plane where each point has the slope:

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}.$$

Example - Construct a phase portrait for the differential equation from the first example.



Critical Point Behavior

The behavior of an autonomous system near an isolated critical point is of particular interest. We'll take a look at some of the most common possibilities.

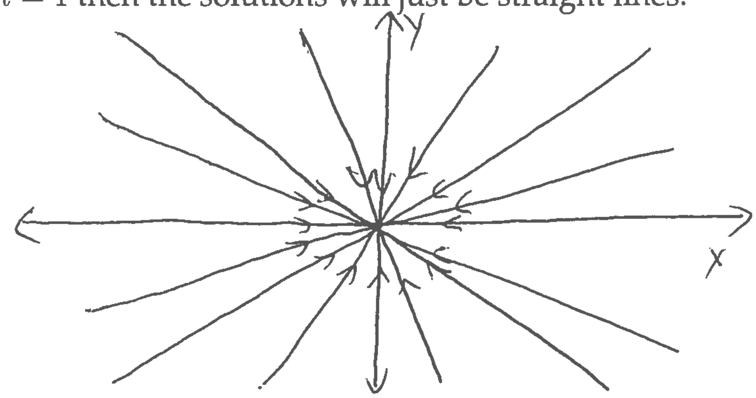
Consider the autonomous linear system:

$$\frac{dx}{dt} = -x,$$
$$\frac{dy}{dt} = -ky.$$

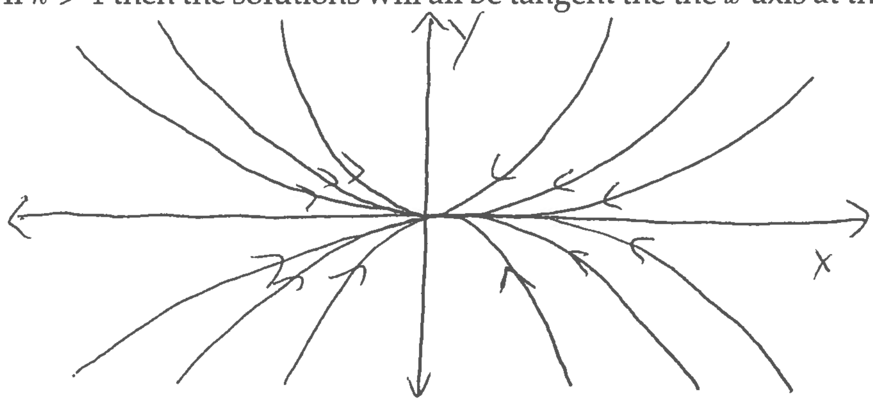
The only critical point for this system is at the origin, and the solution with initial point (x_0, y_0) is:

$$x(t) = x_0 e^{-t}, y(t) = y_0 e^{-kt}.$$

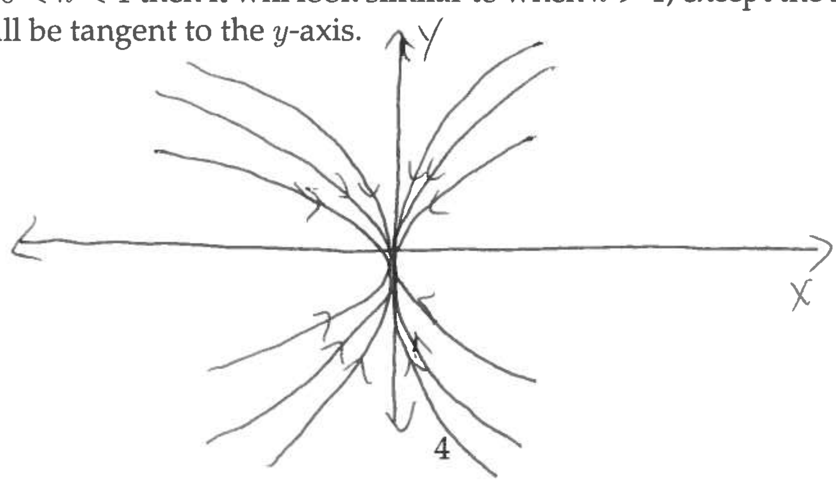
If $k = 1$ then the solutions will just be straight lines:



If $k > 1$ then the solutions will all be tangent the the x-axis at the origin:



If $0 < k < 1$ then it will look similar to when $k > 1$, except the solutions will all be tangent to the y-axis.

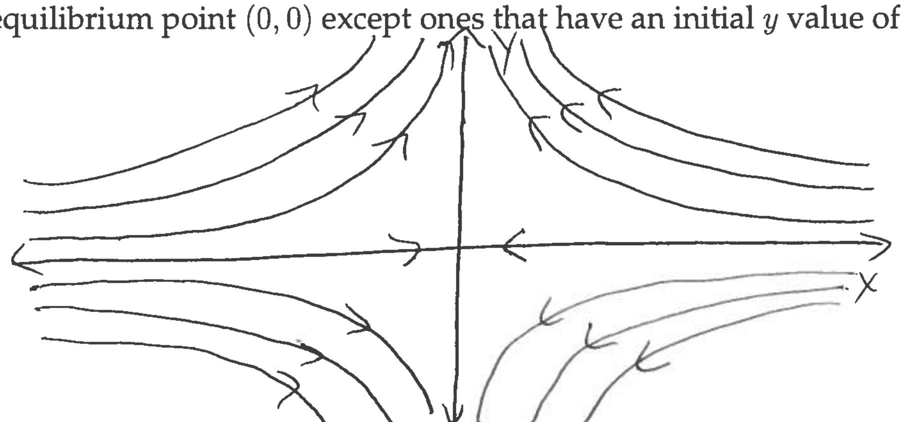


This type of critical point is called a *node*. A critical point is a node provided that:

- Either every trajectory approaches (x_*, y_*) as $t \to \infty$ or every trajectory recedes from (x_*, y_*) as $t \to \infty$.
- Every trajectory is asymptotically tangent to some straight line through the critical point, either as $t \to \infty$ for a sink, or as $t \to -\infty$ for a source. These terms are defined in the next paragraph.

A node is said to be *proper* if no two distinct pairs of "opposite" trajectories are tangent to the same line through the critical point. A node is *improper* if it is not proper. A node is called a *sink* if all trajectories close to it approach it, and a *source* if all trajectories close to it recede from it.

In the above example if $k < 0$ then every trajectory recedes from the equilibrium point $(0, 0)$ except ones that have an initial y value of 0.



This type of critical point is called a *saddle point*.

Stability

A critical point of an autonomous system is said to be *stable* provided that if the initial point is sufficiently close to the critical point, it stays close to the critical point. In mathematical terminology, we say that the point $\mathbf{x}_*$ is stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$|\mathbf{x}_0 - \mathbf{x}_*| < \delta \text{ implies that } |\mathbf{x}(t) - \mathbf{x}_*| < \epsilon$$

for all $t > 0$. A critical point is *unstable* if it is not stable. The earlier example for $k > 0$ had the origin as a stable critical point, while for $k < 0$ the origin was an unstable critical point.

Example - Is the system we examined in the first example a stable system?

Solution - Yes.

Asymptotic Stability

The critical point (x_*, y_*) is called *asymptotically stable* if it is stable and every trajectory that begins sufficiently close to (x_*, y_*) approaches (x_*, y_*) as $t \to \infty$. In mathematical terms we say that there exists a $\delta > 0$ such that:

$$|\mathbf{x}_0 - \mathbf{x}_*| < \delta \text{ implies that } \lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}_*.$$

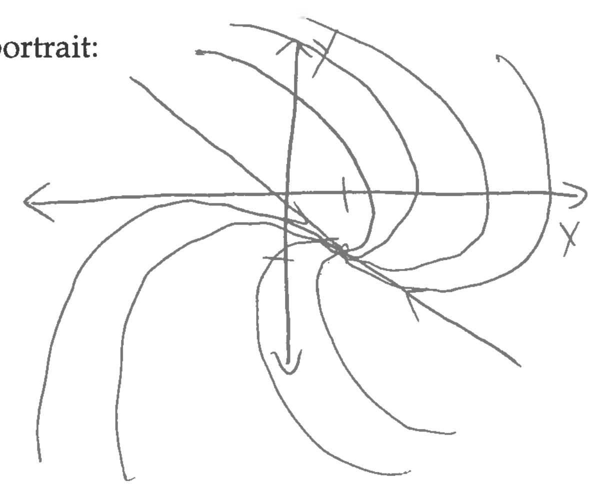
The system from the first example is *not* asymptotically stable.

Example - Construct a phase portrait for the system:

$$\frac{dx}{dt} = 2x - 2y - 4,$$
$$\frac{dy}{dt} = x + 4y + 3,$$

find the critical points, and determine if the critical points are stable points, and if so, whether they are asymptotically stable.

Solution - This system has the phase portrait:

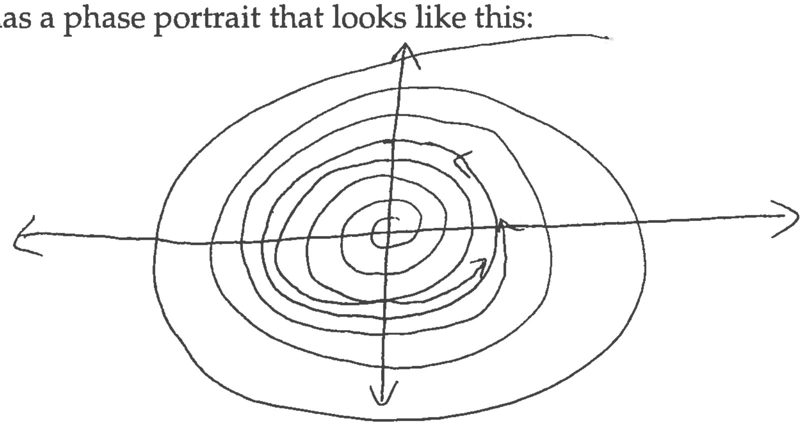


There is only one critical point, the point $(1, -1)$. It is a *sprial point*, which is asymptotically stable.

We've seen examples of trajectories that descend into a critical point, are closed and periodic, and increase without bound. These three possibilities are not the only ones. It's also possible that the solution approaches a closed trajectory. For example, the system:

$$\frac{dx}{dt} = -ky + x(1 - x^2 - y^2),$$
$$\frac{dy}{dt} = kx + y(1 - x^2 - y^2),$$

has a phase portrait that looks like this:



where we see that any trajectory except at the stable point $(0, 0)$ approaches a trajectory around the unit circle.

Now we've seen examples of all the possibilities. In general the four possibilities for a trajectory are:

1. The trajectory approaches a critical point as $t \to \infty$.
2. The trajectory is unbounded with increasing t.
3. The trajectory is a periodic solution with a closed trajectory.
4. The trajectory spirals towards a closed trajectory as $t \to \infty$.

Notes on Homework Problems

Problems 6.1.1 and 6.1.5 ask you to match up systems with phase portraits from the book. Not too hard.

Problems 6.1.10 and 6.1.18 ask you to find equilibrium solutions and then construct phase portraits. You don't need to construct the phase portraits using a computer or graphing calculator. A simple sketch is fine.

Problem 6.1.30 is by far my favorite problem from this section. It steps you through a neat proof tht a solution of an autonomous system either is periodic with a closed trajectory, or else its trajectory never passes through the same point twice. Very interesting.