Math 2280 - Lecture 3: Slope Fields, Existance, and Uniqueness

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In the last lecture we discussed the first-order differential equation in normal form

$$\frac{dy}{dx} = f(x, y)$$

for the special situation where $f(x, y) = f(x)$. In this nice situation there's a well know procedure, namely integration, that we can use for solving the differential equation. Things aren't always so nice. In fact, there exists *no* straightforward procedure by which the general form of this differential equation can be solved explicitly. However, we can use graphical information to construct approximate solutions, and to learn about the behavior of these solutions. Also, even though it might not be easy to find the solution, we can know a solution exists, and that it's unique, if the differential equation satisfies certain conditions.

The assigned problems for this section are:
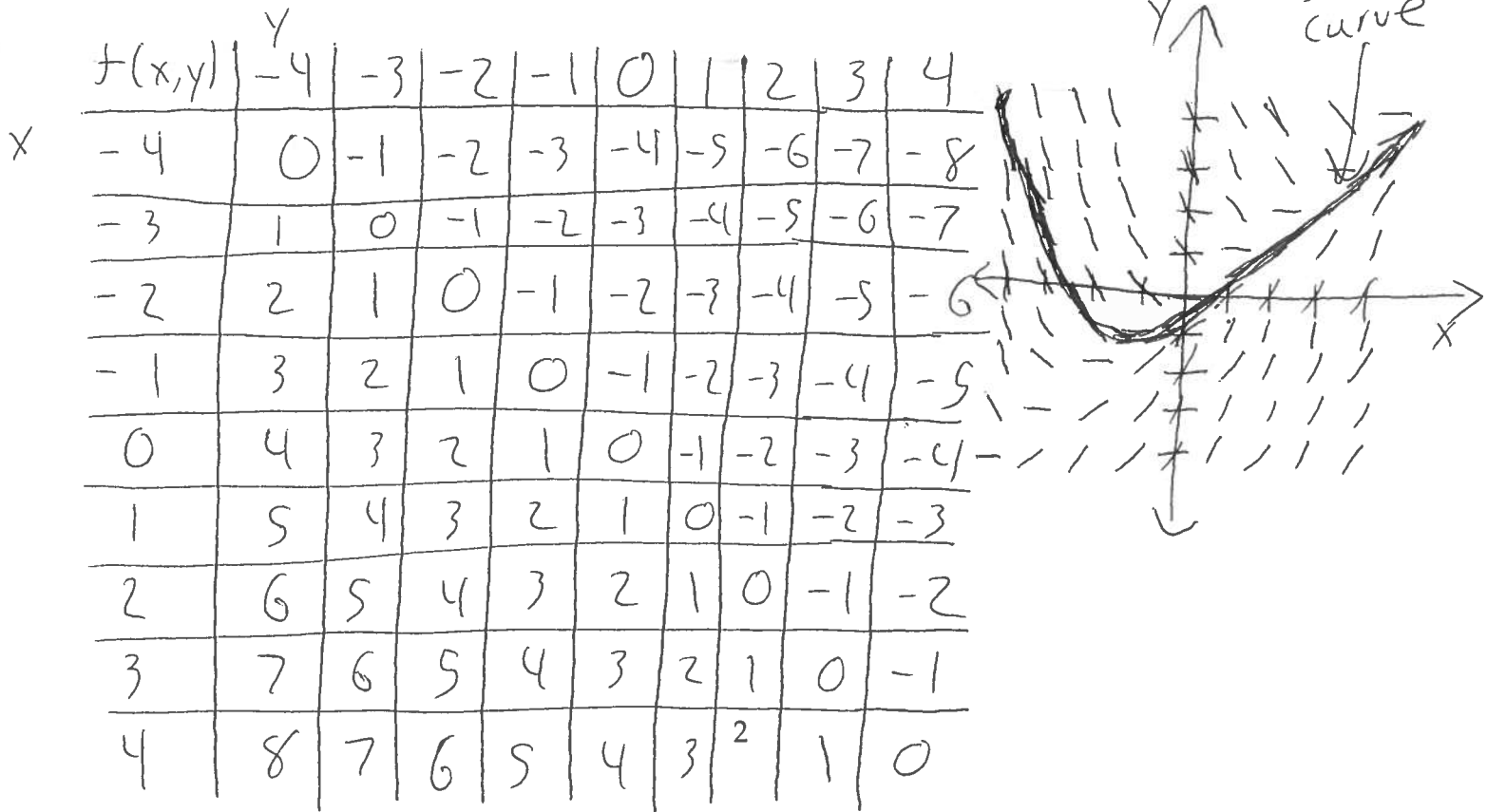
Section 1.3 - 1, 6, 9, 11, 15, 21, 29

Slope Fields

If we're given the differential equation

$$\frac{dy}{dx} = f(x, y)$$

then there's a simple geometric interpretation of the function $f(x, y)$. It tells us the *slope* of the function $y(x)$ at each point. We can use this information to create a *slope field* for the differential equation. A slope field involves a little line segment at a number of points. The tangent line to any solution to the differential equation passing through one of these points must be parallel to the little line segment at that point.

If we're given a point on the curve, we can use this slope field to construct an approximate solution to the differential equation.

Example - Construct a slope field for the differential equation $y' = x - y$ and use it to sketch an approximate solution curve that passes through the point $(-4, 4)$.

$f(x,y)$	-4	-3	-2	-1	0	1	2	3	4
-4	0	-1	-2	-3	-4	-5	-6	-7	-8
-3	1	0	-1	-2	-3	-4	-5	-6	-7
-2	2	1	0	-1	-2	-3	-4	-5	-6
-1	3	2	1	0	-1	-2	-3	-4	-5
0	4	3	2	1	0	-1	-2	-3	-4
1	5	4	3	2	1	0	-1	-2	-3
2	6	5	4	3	2	1	0	-1	-2
3	7	6	5	4	3	2	1	0	-1
4	8	7	6	5	4	3	2	1	0

Solution curve

Now, just by examining this slope field we can deduce a few things. It looks like the line $y = x - 1$ is a solution, and that any other solution curve will asymptotically approach this line as $x \to \infty$.

Existence and Uniqueness of Solutions

Before we spend too long trying to solve a differential equation, we'd like to know if a solution even *exists*, or if we're wasting our time. Also, if we find a solution, it would be nice to know that it's the unique solution.

Now, we're not guaranteed a solution will exist, or that this solution will be unique. For example, the initial value problem

$$y' = \frac{1}{x} \qquad y(0) = 0$$

has *no* solution. The solution curves will all be of the form $\ln |x| + C$, and none of these solution curves is defined at $x = 0$.

On the other hand, the initial value problem

$$y' = 2\sqrt{y} \qquad y(0) = 0$$

has solutions $y_1(x) = x^2$ and $y_2(x) = 0$.

So, is there any way we can know whether a unique solution exists? Yes, in fact, there's a theorem!

Theorem - Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x, y) \qquad y(a) = b$$

has one and only one solution that is defined on the interval I. (This interval might be smaller than the length of the rectangle.)

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As an example where this interval I is smaller than the rectangle, we can examine the initial value problem

$$\frac{dy}{dx} = y^2 \qquad y(0) = 1.$$

The function $f(x, y)$ and its partial derivative with respect to y, namely $2y$, are continuous everywhere. So, they are continuous on the rectangle $-2 < x < 2, 0 < y < 2$. However, the solution

$$y(x) = \frac{1}{1 - x}$$

has a discontinuity at $x = 1$. So, what gives? Well, the interval upon which a solution exists is smaller than the length of the rectangle. What happens is that our solution curve leaves the rectangle, and outside the rectangle existance and uniqueness cannot be guaranteed.

Example - For what initial conditions are we guaranteed a unique solution exists in an interval around the initial point for the differential equation

$$x\frac{dy}{dx} = 2y?$$

Solution - We can rewrite the above differential equation as:

$$\frac{dy}{dx} = \frac{2y}{x}.$$

So, the function $f(x, y)$ from our theorem is:

$$f(x, y) = \frac{2y}{x}.$$

Its partial derivative with respect to y is:

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$$\frac{\partial f}{\partial y} = \frac{2}{x}.$$

Both $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous for $x \neq 0$. So, any initial condition $y(a) = b$ with $a \neq 0$ there will be a unique solution around the point (a, b).

If we're given $y(0) = b$ we are not guaranteed that there exists a unique solution. In fact, all the solutions to the above differential equation will be of the form $y(x) = Cx^2$, for some constant C. If $y(0) = 0$, then there will be an *infinite* number of solutions, as $y(x) = Cx^2$ satisfies this requirement for any value of C. On the other hand, if $y(0) \neq 0$ then there will be *no solution*. So, both of the possible things that can "go wrong", namely that the solution does not exist, or that the solution is not unique, are illustrated here.

Notes On Homework Problems

The first three problems, 1.3.1, 1.3.6, and 1.3.9, ask you to sketch a slope-field. I've scanned the corresponding pages from the textbook and included them in the problem set .pdf on the class website. You can sketch a slopefield on the provided graphs.

The next two problems, 1.3.11 and 1.3.15, are straightforward existence and uniqueness problems similar to the examples given above. Remember, you can't divide by zero!

On problem 1.3.21 you're asked to sketch a slopefield, and then use it to estimate a solution curve satisfying a given initial condition. Be sure to graph this one carefully!

Problem 1.3.29 demonstrates how you can have many different solutions to the same differential equations, and is a good example of when and where you know a unique solution to an initial value problem exists. Probably the most interesting problem from this section.