# Math 2280 - Lecture 34

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In the last lecture we learned how to solve linear ODEs of the form:

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

around ordinary points using power series. Today, we'll learn how to solve them around a specific type of singular point called a *regular singular point*.

Today's lecture corresponds with section 8.3 from the textbook. The assigned problems are:

Section 8.3 - 1, 8, 15, 18, 24

## Regular Singular Points

Last time we looked at how to solve linear ODEs of the form:

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

The first thing we do is rewrite the ODE as:

$$y'' + P(x)y' + Q(x)y = 0,$$

where, of course,

$$P(x) = \frac{B(x)}{A(x)}, \text{ and } Q(x) = \frac{C(x)}{A(x)}.$$

If $P(x)$ and $Q(x)$ are analytic around the point $a$ then we know there are two linearly independent solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

where the radii of convergence are at least as great as the distance in the complex plane from $a$ to the nearest singular point of either $P(x)$ or $Q(x)$.

## Ordinary, Regular, and Irregular Points

We first state without proof that either $P(x)$ and $Q(x)$ are analytic at $x = a$ or approach $\pm\infty$ as $x \to a$.

Now, of course, we must ask what we do if either $P(x)$ or $Q(x)$ is not analytic at $a$. So, what do we do? It turns out we have methods for dealing with this as long as they fail to be analytic in the "right way". We'll get into what that means in just a moment.

We'll restrict ourselves to dealing with the case $a = 0$, but we note that by just shifting our coordinates this restriction incurs no loss of generality.

Alright. So, we divide singular points into two types: regular singular points, and irregular singular points. A regular singular point is a singular point where, if we rewrite:

$$y'' + P(x)y' + Q(x)y = 0$$

as

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0$$

the functions $p(x)$ and $q(x)$ are analytic. This is the situation upon which we'll focus. We will not discuss how to solve ODEs around irregular singular points, as that is a much more difficult and advanced topic.

*Example* - Determine whether $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point of the ODE:

$$x^2 y'' + (6 \sin x) y' + 6y = 0$$

*Solution* - If we divide through by $x^2$ we get:

$$y'' + \frac{6 \sin x}{x^2} y' + \frac{6}{x^2} y = 0$$

which has $\lim_{x \to 0} P(x) = \lim_{x \to 0} Q(x) = \infty$, so $x = 0$ is a singular point. If we rewrite this in the format above we have:

$$p(x) = \frac{6 \sin x}{x}, \text{ and } q(x) = 6,$$

both of which are analytic at $x = 0$. So, $x = 0$ is a regular singular point of the ODE.

We again state a fact without proof. If the limits:

$$\lim_{x \to 0} p(x) \quad \text{and} \quad \lim_{x \to 0} q(x)$$

exist, are finite, and are not both $0$ then $x = 0$ is a regular singular point. If both limits are $0$ then $x = 0$ may be a regular singular point or an ordinary point. If either limit fails to exists or is $\pm\infty$ then $x = 0$ is an irregular singular point. This gives us a useful way for testing if a singular point is regular.

## The Method of Frobenius

Now we'll figure out how to actually solve these ODEs around regular singular points. We start by examining the simplest such ODE:

$$x^2 y'' + p_0 x y' + q_0 y = 0$$

where $p_0, q_0$ are both constants. This ODE is solved by $y = x^r$, where $r$ satisfies the quadratic:

$$r(r-1) + p_0 r + q_0 = 0.$$

Using this as our starting point, in general we assume our solution has the form:

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n.$$[1]

This is called a Frobenius series. We want to figure out what this constant $r$ needs to be. So, assume that we have a solution in this form. In this case we have:

[1]This is *NOT* a power series if $r \notin \mathbb{Z}^+$.

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r},$$

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1},$$

$$y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}.$$

We substitute these into:

$$x^2 y'' + xp(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are analytic around $x = 0$, and so have a power series representation of the form:

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots$$
$$q(x) = q_0 + q_1 x + q_2 x^2 + \cdots.$$

Plugging all this stuff in we get:

$$\begin{gathered}[r(r-1)c_0 x^r + (r+1)rc_1 x^{r+1} + \cdots] \\ +[p_0 x + p_1 x^2 + \cdots] \cdot [rc_0 x^{r-1} + (r+1)c_1 x^r + \cdots] \\ +[q_0 + q_1 x + \cdots] \cdot [c_0 x^r + c_1 x^{r+1} + \cdots] = 0.\end{gathered}$$

If we examine the $x^r$ term we get, assuming (as we of course always can and should) that $c_0 \neq 0$, we get the relation:

$$r(r-1) + p_0 r + q_0 = 0.$$

This is called the indicial equation of the ODE, and it must, according to the identity principle, be satisfies for our solution to work. This is, of course, only a necessary condition, and we certainly haven't proven it's sufficient. That's where the next theorem comes in:

**Theorem** - Suppose that $x = 0$ is a regular singular point of the ODE:

$$x^2y'' + xp(x)y' + q(x)y = 0.$$

Let $\rho > 0$ denote the minimum of the radii of convergence of the power series:

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \text{ and } q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Let $r_1$ and $r_2$ be the real roots (we'll always be assuming our roots are real), of the indicial equation with $r_1 \geq r_2$. Then

1. For $x > 0$, there exists a solution to our ODE of the form:

   $$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0,$$

   corresponding to the larger root $r_1$.

2. If $r_1 - r_2$ is neither zero nor a positive integer, then there exists a second linearly independent solution for $x > 0$ of the form:

   $$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n, \quad b_0 \neq 0,$$

   corresponding to the smaller root $r_2$.

The radii of convergence of $y_1$ and $y_2$ are at least $\rho$. We determine the coefficients by plugging our series into:

$$x^2y'' + xp(x)y' + q(x)y = 0.$$

*Example* - Use the method of Frobenius to solve the ODE:

$$2x^2y'' + 3xy' - (x^2+1)y = 0$$

around the regular singular point $x = 0$.

*Solution* - Rewriting this we have:

$$y'' + \frac{\frac{3}{2}}{x}y' + \frac{-\frac{1}{2} - \frac{1}{2}x^2}{x^2}y = 0$$

and so $p_0 = \frac{3}{2}$ and $q_0 = -\frac{1}{2}$.

This gives us the indicial equation:

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = (r - \frac{1}{2})(r+1) = 0,$$

and so our two roots are $r_1 = \frac{1}{2}$ and $r_2 = -1$. So, our theorem guarantees two linearly independent Frobenius type solutions.

Frequently it's easier to work out our solutions without plugging in specific values of $r$ until the end. That's what we'll do here. Now, if we have a solution of the form:

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r},$$

then if we plug this form into our ODE we get the relation:

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r} + 3\sum_{n=0}^{\infty}(n+r)x^{n+r} - \sum_{n=0}^{\infty}c_nx^{n+r+2} - \sum_{n=0}^{\infty}c_nx^{n+r} = 0.$$

If we shift the third series over by $2$ we get:

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + 3\sum_{n=0}^{\infty}(n+r)x^{n+r} - \sum_{n=2}^{\infty} c_{n-2}x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

From the $x^r$ and $x^{r+1}$ coefficients we get the relations:

$$[2r(r-1)+3r-1]c_0 = 2(r^2+\frac{1}{2}r-\frac{1}{2})c_0 = 0$$
$$[2(r+1)r+3(r+1)-1]c_1 = 0.$$

If we plug in our values for $r$ we see that the first of these is automatically satisfies for any $c_0$, as the multiplier of $c_0$ is just a constant multiplied by the indicial equation. On the other hand, if we plug in our values for $r$ we see that the second equation is only satisfied for $c_1 = 0$.

As for the other coefficients we get the relations:

$$2(n+r)(n+r-1)c_n + 3(n+r)c_n - c_{n-2} - c_n = 0,$$

which simplify to:

$$c_n = \frac{c_{n-2}}{2(n+r)^2+(n+r)-1} \text{ for } n \geq 2.$$

So, given $c_1 = 0$, all the odd coefficients must be $0$. As for the even coefficients, for $r = \frac{1}{2}$ we get:

$$a_n = \frac{a_{n-2}}{2n^2+3n},$$

and for $r = -1$ we get:

$$b_n = \frac{b_{n-2}}{2n^2-3n}.$$

And, well, that's pretty much as good as we can do. If we write out our first few terms we get:

$$y_1(x) = a_0 x^{\frac{1}{2}} \left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \frac{x^6}{55,440} + \cdots\right),$$

and

$$y_2(x) = b_0 x^{-1} \left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \frac{x^6}{2160} + \cdots\right).$$

# Notes on Homework Problems

Problems 8.3.1 and 8.3.8 are only asking you to determine the type of point (ordinary, regular singular, or irregular singular) $x = 0$ is, and if it's a regular singular point, determine the roots of the indicial equation. These problems are NOT asking you to solve the given differential equations.

Problem 8.3.15 is essentially the same type of problem as 8.3.1 and 8.3.8, just around a point different than $x = 0$.

Problems 8.3.18 and 8.3.24 actually want you to solve some differential equations. These can be kind of long, and it might take a little while to get your solutions into the form at the back of the book. Please note, if you're comparing your solution with the back of the textbook, that the notation $(2n+1)!!$ means the product of the first $n$ odd terms, it does *not* mean you take the factorial twice!