Math 2280 - Lecture 33

Dylan Zwick

Fall 2013

Today we're going to examine how we find power series solutions to ordinary differential equations of the form

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Now, as we saw in the second example from our last lecture, we're not always able to find a solution to a differential equation of this form. Today, we're going to focus on a situation where we know we can find a solution. In particular, around a point a where $P(x) = B(x)/A(x)$, and $Q(x) = C(x)/A(x)$ are "analytic". Such a point a is called an *ordindary point*.

Today's lecture corresponds with section 8.2 from the textbook. The assigned problems for this section are

Section 8.2 - 1, 7, 14, 17, 32

Series Solutions Near Ordinary Points

Recall that a function $f(x)$ is analytic at a point $x = a$ if $f(x)$ is equal to a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all values of x around a.

Suppose we have a second-order linear differential equation of the form

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

where $A(x), B(x)$, and $C(x)$ are analytic at $x = a$. If the functions $P(x) = B(x)/A(x)$ and $Q(x) = C(x)/A(x)$ are also analytic at $x = a$, then the differential equation has two linearly independent power series solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

The radius of convergence of any such series is at least as large as the distance from a to the nearest (real or complex) singular point of the differential equation.

Note that polynomials are always analytic. If $A(x), B(x)$, and $C(x)$ are polynomials, then $P(x) = B(x)/A(x), Q(x) = C(x)/A(x)$ will be analytic at $x = a$ if $A(a) \neq 0$.

Now, it could be the case that $A(a) = 0$, but $P(x)$ is *still* analytic at $x = a$. For example,

$$P(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

is considered analytic at $x = 0$.[1] But, for most of the problems we'll be dealing with, the functions $A(x), B(x)$, and $C(x)$ will be polynomials, and the only possible problem points are where $A(x) = 0$.

[1] The book doesn't do a great job of explaining this. Really, the function $B(x)/A(x)$ would not be defined at $x = a$, and so would *not* be analytic at $x = a$. However, there could be a well-defined value for $P(a)$ that would make $P(x)$ analytic at $x = a$, and so we essentially "fill in" that point. Yeah, it's confusing, and explaining exactly what's going on in detail would take us too far afield, so don't worry too much about it for now. If you take a more advanced class, it will be explained.

Example - Find a general solution in powers of x to the differential equation

$$(x^2 + 2)y'' + 4xy' + 2y = 0.$$

More room for example problem.

Example - Find a general solution in powers of x to the differential equation

$$y'' + x^2 y' + 2xy = 0.$$

More room for example problem.

Notes on Homework Problems

Problems 8.2.1, 8.2.7, and 8.2.14 are all problems in the same vein as the example problems from these lecture notes. Good for practice. Problem 8.2.17 is similar, only initial conditions are specified, so the initial coefficients for the power series will be determined.

Problem 8.2.32 is more complicated. It walks you through a proof for establishing a famous formula used in calculating polynomials called Legendre polynomials. This is a worthwhile problem, but it's kind of long, and won't be graded, so make sure you've got the first four problems before you tackle this one.