Math 2280 - Lecture 32

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So far in this course we've focused almost exclusively on solving linear differential equations with constant coefficients. But these are, to say the least, not all the differential equations that are out there. For example, a differential equation that is encountered very frequently in applied mathematics is *Bessel's equation* of order n:

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

This is a linear differential equation, but its coefficient functions are definitely not all constants. We'd like a method for solving this type of differential equation.

Such a method exists, and it's very powerful. It involves representing the solution as a power series[1], and then figuring out what this power series is. As I said, this method is very powerful, but it can require some work. Fair warning.

This lecture corresponds with section 8.1 from the textbook. The assigned problems are:

Section 8.1 - 2, 8, 13, 21, 25

[1]Actually, for Bessel's equation, a Frobenius series. But, we'll get into that later.

Introduction and Review of Power Series

A power series is an infinite series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

If $a = 0$ then we call it a power series in x:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

We will confine ourselves mainly to power series in x, but every general property of power series in x can be converted to a general property of power series in $(x-a)$.

We say a power series converges on the interval I provided that the limit

$$\sum_{n=0}^{\infty} c_n x^n = \lim_{N\to\infty} \sum_{n=0}^{N} c_n x^n$$

is defined for all $x \in I$. In this case the sum

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

is defined on I, and we call the series a *power series representation* of the function f on I.

Some common power series representations are:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

The first two series converge for all x, while the third, called the geometric series, only converges for $|x| < 1$.

The Power Series Method

The *power series method* for solving a differential equation consists of substituting the power series

$$y = \sum_{n=0}^{\infty} c_n x^n$$

into the differential equation, and then attempting to determine what the coefficients $c_0, c_1, c_2, \ldots$ must be in order for the power series to satisfy the differential equation.

In solving these differential equations, there are two very important theorems:

Theorem - *Termwise Differentiation of Power Series*

If the power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

of the function f converges on the open interval I, then f is differentiable on I, and

$$f'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

at each point of I.

The other important theorem is:

Theorem - *Identity Principle*

If

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for every point x in some open interval I, then $a_n = b_n$ for all $n \geq 0$.

In particular, if $\sum a_n x^n = 0$ for all x in some open interval, it follows from the identity principle that $a_n = 0$ for all $n \geq 0$.

Now, if we have a power series solution to a differential equation, an important question is the interval upon which the series converges. A useful test for determining this interval is the following:

Theorem - *Radius of Convergence*

Given the power series $\sum c_n x^n$, suppose that the limit

$$\rho = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

exists (ρ is finite) or is infinite. Then

(a) If $\rho = 0$, then the series diverges for all $x \neq 0$.

(b) If $0 < \rho < \infty$, then $\sum c_n x^n$ converges if $|x| < \rho$ and diverges if $|x| > \rho$.

(c) If $\rho = \infty$, then the series converges for all x.

The number ρ is called the *radius of convergence* of the power series $\sum c_n x^n$.

Let's see how the power series method works with a few examples.

Example - Solve the differential equation $y' = y$.

Solution - If we make the substitution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

we get the relation

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} c_n x^n.$$

We can rewrite this as:

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0.$$

If we shift the sum on the left by 1 we can combine the two sums to get:

$$\sum_{n=0}^{\infty} ((n+1)c_{n+1} - c_n)x^n = 0.$$

The identity principle tells us we must have the relations

$$(n+1)c_{n+1} - c_n = 0.$$

So, we have the recurrence relation

$$c_{n+1} = \frac{c_n}{n+1}.$$

The first few terms are:

$$\begin{aligned} c_0 &= c_0, \\ c_1 &= \frac{c_0}{1}, \\ c_2 &= \frac{c_1}{2} = \frac{c_0}{1 \times 2}, \\ c_3 &= \frac{c_2}{3} = \frac{c_0}{1 \times 2 \times 3} = \frac{c_0}{3!}, \end{aligned}$$

and, in general, $c_n = \dfrac{c_0}{n!}$.

So,

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x.$$

But, we already knew that, didn't we!

Example - Solve the differential equation $x^2 y' = y - x - 1$.

Solution - Again, we substitute the solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

into the differential equation. Doing this gives us the relation

$$\sum_{n=1}^{\infty} n c_n x^{n+1} = (c_0 - 1) + (c_1 - 1)x + \sum_{n=2}^{\infty} c_n x^n.$$

The coefficients in front of x^k for all k must be equal, and so we get $c_0 = c_1 = 1$, and the series equality

$$\sum_{n=1}^{\infty} n c_n x^{n+1} - \sum_{n=2}^{\infty} c_n x^n = 0.$$

If we shift the sum on the left by 1 and the sum on the right by 2 we get

$$\sum_{n=0}^{\infty}((n+1)c_{n+1} - c_{n+2})x^{n+2} = 0.$$

So, this gives us $c_{n+2} = (n+1)c_{n+1}$. The first few terms are:

$$\begin{aligned} c_2 &= 1 \cdot c_1 = c_1, \\ c_3 &= 2 \cdot c_2 = (2 \times 1)c_1, \\ c_4 &= 3 \cdot c_3 = (3 \times 2 \times 1)c_1, \end{aligned}$$

and, in general, $c_n = (n-1)!c_1$.

So, as $c_1 = 1$, our solution is

$$y(x) = 1 + x + \sum_{n=2}^{\infty}(n-1)!x^n.$$

Hmmm... something fishy here. Let's look at the radius of convergence for this series.

$$\lim_{n\to\infty}\left|\frac{(n-1)!}{n!}\right| = \lim_{n\to\infty}\frac{1}{n} = 0.$$

So, the series diverges(!) for all values of x outside $x = 0$. What does this mean? It means our differential equation does not have a convergent power series solution of the assumed form.[2] Lesson - always check for convergence.

[2] Not too surprising, as the differential equation $y' - \frac{y}{x^2} + \frac{x+1}{x^2} = 0$ is not defined at $x = 0$.

Example - Solve the differential equation $y'' + y = 0$.

Solution - Yet again, we make the substitution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Making this substitution we get the equation

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Shifting the sum on the left by 2 we get the relation

$$\sum_{n=0}^{\infty}((n+2)(n+1)c_{n+2} + c_n)x^n = 0.$$

From the identity principle this gives us

$$c_{n+2} = -\frac{c_n}{(n+1)(n+2)}.$$

The terms will break up into odd and even parts[3], and the relations we'll get are:

$$c_{2k} = \frac{(-1)^k c_0}{(2k)!},$$

and

$$c_{2k+1} = \frac{(-1)^k c_1}{(2k+1)!}.$$

So, our solution will be:

[3]Just as we've done before, just take the first few terms and look for patterns...

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

We recognize the first summation as the cosine function, and the second summation as the sine function. In fact, this is how we could *define* the cosine and sine functions, in terms of the power series that satisfies a given differential equation with some set initial conditions. This is, in fact, how many famous functions in applied mathematics come about.

Notes on Homework Problems

Problems 8.1.2, 8.1.8, and 8.1.13 ask you to solve differential equations using the power series method. Pretty much what we've done in the example problems from these notes.

Problem 8.1.21 is similar to the first three problems, and similar to the example problems from these notes, except you're asked to take into account specified initial conditions. So, the unknown constants take on known values.

Problem 8.1.25 is lots of fun. It also, although it doesn't mention this in the property, relates to a connection between the Fibonacci numbers are the famous "golden ratio". You may remember this from The Da Vinci Code.