

Math 2280 - Lecture 2: How To Use Integration To Solve Differential Equations

Dylan Zwick

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The major topic of calculus II was the study of integrals and how we integrate functions. Well, it turns out that finding an integral is actually solving a special type of differential equation.

We're going to talk today about the form of these differential equations, how they come up, and how we can extract some more general ideas about differential equations based upon these (relatively) simple ones.

Today's lecture corresponds with section 1.2 from the textbook. The assigned problems from this section are:

Section 1.2 - 1, 6, 11, 15, 27, 35, 43

Note that I've assigned a fair number of these problems because they're pretty easy and it's good practice for harder stuff later on.

Integrate Your Way to Wealth and Happiness, or at Least a Solution

If we have a first-order differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

then it takes a particularly simple form if $f(x, y)$ can be written as just a function of x , so $f(x)$. Then we have an equation of the form

$$\frac{dy}{dx} = f(x).$$

To solve this for $y(x)$ we just need to integrate both sides of the equation

$$y(x) = F(x) + C$$

where C is an arbitrary constant and $F(x)$ is a function such that $F'(x) = f(x)$. We usually choose $F(x)$ so that $F(0) = 0$.

If all we're given is a differential equation, then this is as far as we can go. However, if we're given the value of $y(x)$ at a point, say $y(x_0) = y_0$, then we can solve for C :

$$C = y_0 - F(x_0)$$

to get a unique solution to our *initial value problem*. It's the one and only solution satisfying our differential equation *and* $y(x_0) = y_0$.

Example - Find a function satisfying the differential equation

$$\frac{dy}{dx} = (x - 2)^2$$

with the initial condition $y(2) = 1$.

Solution - The antiderivative of $(x - 2)^2$ is $\frac{(x - 2)^3}{3}$, and therefore

$$y(x) = \frac{(x - 2)^3}{3} + C.$$

If we plug in $x = 2$ we get $y(2) = C$. So, if $y(2) = 1$ we have $C = 1$, and a function satisfying the above differential equation with the given initial condition is:

$$y(x) = \frac{(x-2)^3}{3} + 1.$$

If we have a second-order differential equation of the form

$$\frac{d^2y}{dx^2} = f(x)$$

then if we integrate both sides of the equation once we get

$$\frac{dy}{dx} = F(x) + C_1,$$

and if we integrate again we get

$$y(x) = \int F(x)dx + C_1x + C_2.$$

This solution has *two* unknown constants. If we know $y(x_0)$ and $y'(x_0)$ then we can solve for C_1 and C_2 . So, for a second-order differential equation there are two unknown constants, and we need two initial conditions.

This is a general phenomenon. We'll find that, generally speaking, the solution to an n -th order differential equation will require n initial conditions to specify a particular solution.

Example - What is the solution to the differential equation

$$\frac{d^2y}{dx^2} = 3$$

if $y'(2) = 4$ and $y(2) = 7$?

Solution - If we integrate the above equation once we get:

$$y'(x) = 3x + C_1.$$

We know $y'(2) = 4$, and so $4 = 3(2) + C_1$, which means $C_1 = -2$. This gives us the equation

$$\frac{dy}{dx} = 3x - 2.$$

Integrating this equation we get:

$$y(x) = \frac{3}{2}x^2 - 2x + C_2.$$

We know $y(2) = 6 - 4 + C_2 = 7$, and so $C_2 = 5$. Therefore, our solution is:

$$y(x) = \frac{3}{2}x^2 - 2x + 5.$$

Velocity and Acceleration

Newton's second law of motion states that the net force upon an object is equal to the object's mass multiplied by its acceleration. Acceleration is the rate of change of velocity, which is itself the rate of change of position.

So, if we represent the acceleration of an object at time t as $a(t)$, the velocity at time t as $v(t)$, and the position at time t as $x(t)$, then we have the relations;

$$\frac{dx}{dt} = v(t),$$

$$\frac{dv}{dt} = a(t),$$

and therefore

$$\frac{d^2x}{dt^2} = a(t).$$

So, acceleration is the second derivative of position with respect to time.

If an object is experiencing constant acceleration a then

$$a(t) = a,$$

and therefore its velocity will be

$$v(t) = \int a = at + C_1.$$

If the velocity at time $t = 0$ is v_0 , then we have

$$v(t) = at + v_0.$$

If we then integrate this again to get position we get

$$x(t) = \frac{1}{2}at^2 + v_0t + C_2.$$

If the position at time $t = 0$ is x_0 , then our position function is

$$x(t) = \frac{1}{2}at^2 + v_0t + x_0.$$

Example - At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point C , 35 miles away. If the constantly accelerated car arrives at C with a velocity of 60 mi/h, at what time does it arrive at C ?

Solution - The car moves at a constant acceleration a , and when it arrives at C its velocity is 60 mi/h. So, if we set $t = 0$ to be noon, and say t_* is the arrival time, we have:

$$v(t_*) = at_* = 60.$$

So, we have $a = \frac{60}{t_*}$. Now, if we say $x = 0$ at point A , then the total distance traveled at time t_* will be:

$$x(t_*) = \frac{1}{2}at_*^2 = 35.$$

If we plug in $a = \frac{60}{t_*}$ we get:

$$30t_* = 35,$$

and so

$$t_* = \frac{35}{30} = \frac{7}{6} \text{ hours.}$$

So, the arrival time is 1:10 PM.

Notes On The Homework Problems

The first five problems (1.2.1, 1.2.6, 1.2.11, 1.2.15, 1.2.27) are all either simple integration problems or simple kinematics problems. Shouldn't be too bad.

The sixth problem (1.2.35) asks you to derive a formula for how fast a stone hits the ground when it's dropped from a given height. Not too hard, but fun and interesting to do.

The seventh problem (1.2.43) is for you science fiction fans. Note that the answer you get should be a very, very long distance. However, almost all the stars you see in the night sky are much farther away than your answer!