Math 2280 - Lecture 28

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Today we're going to delve deeper into how to calculate inverse Laplace transforms. In particular, we're going to discuss methods for calculating inverse Laplace transforms for rational functions, which are functions of the form:

$$\frac{P(s)}{Q(s)},$$

where both $P(s)$ and $Q(s)$ are polynomials. We'll assume the degree of $P(s)$ is less than the degree of $Q(s)$. As $\lim_{s\to\infty} F(s) = 0$ for any Laplace transform $F(s)$, this assumption should always be valid.

This lecture corresponds with section 7.3 from the textbook. The assigned problems are:

Section 7.3 - 3, 8, 19, 24, 30, 33

Translation and Partial Fractions

As mentioned in the introduction, today we're going to focus on how to find inverse Laplace transforms for functions of the form:

$$R(s) = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are polynomials, and the degree of $P(s)$ is less than that of $Q(s)$. The idea is that we want to use a partial fraction decomposition. In other words, we want to factor $Q(s)$ into its linear and quadratic parts (which can always, in theory, be done for polynomials with real coefficients) and then based upon these parts write our quotient as a sum according to two rules:

Rule 1 - Linear Factor Partial Fractions

> The portion of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $(s-a)$ (where we mean a linear factor of $Q(s)$) of multiplicity n is a sum of n partial fractions, having the form:
>
> $$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$
>
> where $A_1, A_2, \ldots, A_n$ are constants.

Rule 2 Quadratic Factor Partial Fractions

> The portion of the partial fraction decomposition corresponding to the irreducible quadratic factor $(s-a)^2 + b^2$ of multiplicity m is a sum of m partial fractions, having the form:
>
> $$\frac{A_1 s + B_1}{(s-a)^2 + b^2} + \frac{A_2 s + B_2}{[(s-a)^2 + b^2]^2} + \cdots + \frac{A_m s + B_m}{[(s-a)^2 + b^2]^m},$$
>
> where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants.

After we find the partial fraction decomposition we then use it to find the inverse Laplace transform.

The formulas and relations that we're going to find most useful are:

Theorem - If $F(s) = \mathcal{L}(f(t))$ exists for $s > c$, then $\mathcal{L}(e^{at}f(t))$ exists for $s > a + c$, and

$$\mathcal{L}(e^{at}f(t)) = F(s-a),$$

or equivalently

$$\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t)$$

Proving this is very simple:

$$F(s-a) = \int_0^\infty e^{-(s-a)t}f(t)dt = \int_0^\infty e^{-st}[e^{at}f(t)]dt = \mathcal{L}(e^{at}f(t)).$$

This fact, combined with the relations:

$$\mathcal{L}(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}, \quad s > a;$$

$$\mathcal{L}(e^{at}\cos kt) = \frac{s-a}{(s-a)^2+k^2}, \quad s > a;$$

$$\mathcal{L}(e^{at}\sin kt) = \frac{k}{(s-a)^2+k^2}, \quad s > a;$$

$$\mathcal{L}\left(\frac{e^{at}}{2k}t\sin kt\right) = \frac{s-a}{((s-a)^2+k^2)^2}, \quad s > a;$$

$$\mathcal{L}\left(\frac{e^{at}}{2k^3}(sinkt - kt\cos kt)\right) = \frac{1}{((s-a)^2+k^2)^2} \quad s > a;$$

will allow us to figure out the inverse Laplace transform given almost any partial fraction decomposition. I say *almost* any because you might have a quadratic term to a higher than second power. In this case I'd say, first, that you shouldn't see those in this class, and second we'll go over how you can calculate those Laplace transforms using convolutions in the next lecture. So, we'll see that, in theory, the inverse Laplace transform for any partial fraction decomposition can be calculated. It just might take a while. Note that a repeated quadratic factor usually corresponds to a situation where we have resonance in our system.

Example - Solve the initial value problem:

$$y'' + 4y' + 4y = t^2; y(0) = y'(0) = 0.$$

Solution - Now, we could solve this using techniques we already know, namely the method of undetermined coefficients, but let's figure this out using Laplace transforms. If we take the Laplace transform of both sides we get:

$$s^2Y(s) + 4sY(s) + 4Y(S) = \frac{2}{s^3}.$$

If we then solve this for $Y(s)$ we get:

$$Y(s) = \frac{2}{s^3(s^2 + 4s + 4)} = \frac{2}{s^3(s+2)^2}.$$

If we calculate the partial fraction decomposition of this we know it will be of the form:

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s+2)} + \frac{E}{(s+2)^2}.$$

If we then solve this for the unknowns (equate the coefficients, do some linear algebra...) we get:

$$Y(s) = \frac{3/8}{s} - \frac{1/2}{s^2} + \frac{1/2}{s^3} - \frac{3/8}{s+2} - \frac{1/4}{(s+2)^2}.$$

This will have the inverse Laplace transform:

$$y(t) = \frac{3}{8} - \frac{1}{2}t + \frac{1}{4}t^2 - \frac{3}{8}e^{-2t} - \frac{1}{4}te^{-2t}$$

which is our desired solution. Sweet!

Example - Apply the translation formula to find the Laplace transform of:

$$f(t) = e^{-2t} \sin 3\pi t$$

Solution - We just look at our table to get:

$$\mathcal{L}(f(t)) = \frac{3\pi}{(s+2)^2 + 9\pi^2}.$$

Example - Calculate the inverse Laplace transform of the function:

$$F(s) = \frac{s-1}{(s+1)^2}$$

Solution - It becomes more straightforward if we rewrite $F(s)$ as:

$$\frac{s-1}{(s+1)^2} = \frac{s+1}{(s+1)^2} - \frac{2}{(s+1)^2} = \frac{1}{s+1} - \frac{2}{(s+1)^2}.$$

Then, we can just read the inverse transforms from our list of standard inverse transforms:

$$f(t) = e^{-t} - 2te^{-t} = (1-2t)e^{-t}.$$

Example - Calculate the inverse Laplace transform of the function:

$$F(s) = \frac{1}{(s^2 + s - 6)^2}$$

Solution - If we factor the denominator we get:

$$F(s) = \frac{1}{(s+3)^2(s-2)^2}$$

which will have the partial fraction decomposition:

$$\frac{1}{(s+3)^2(s-2)^2} = \frac{A_1}{s+3} + \frac{A_2}{(s+3)^2} + \frac{A_3}{s-2} + \frac{A_4}{(s-2)^2}.$$

This would then imply:

$$1 = A_1(s+3)(s-2)^2 + A_2(s-2)^2 + A_3(s+3)^2(s-2) + A_4(s+3)^2.$$

If we plug in $s = 2$ we get $A_4 = \frac{1}{25}$. If we plug in $s = -3$ we get $A_2 = \frac{1}{25}$. After some algebra we can get $A_1 = \frac{2}{125}$ and $A_3 = -\frac{2}{125}$. Plugging in these values we get:

$$\frac{1}{(s+3)^2(s-2)^2} = \frac{\frac{2}{125}}{s+3} + \frac{\frac{1}{25}}{(s+3)^2} - \frac{\frac{2}{125}}{(s-2)} + \frac{\frac{1}{25}}{(s-2)^2}.$$

From here we can again read off the inverse Laplace transform from our table in a straightforward way:

$$f(t) = \frac{2}{125}e^{-3t} + \frac{1}{25}te^{-3t} - \frac{2}{125}e^{2t} + \frac{1}{25}te^{2t}.$$

Notes on Homework Problems

Problems 7.3.3 and 7.3.8 are straightforward applications of the translation theorem, one involving finding a Laplace transform, the other involving finding an inverse Laplace transform.

Problem 7.3.19 gives you some experience with partial fraction decomposition. Problem 7.3.24 is a slightly more difficult partial fraction decomposition problem, but fortunately the hardest factorization is done for you.

problem 7.3.30 and 7.3.33 are both linear differential equations that you should solve using Laplace transform methods.