

Math 2280 - Lecture 25

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So far in our study of how to solve systems of ODEs we've focused almost exclusively on systems of the form:

$$\mathbf{x}' = A\mathbf{x}.$$

This is just the homogeneous case. It's an extremely important case, but it can also be viewed as, in some sense, the easiest possible case! In general, we'd like to be able to solve a system of ODEs of the form:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$$

for *any* value of the function $\mathbf{f}(t)$. How do we do this? Well, the same way we did it in chapter 3. We guess! In particular, if our function $\mathbf{f}(t)$ is a combination (sums and products) of polynomials, exponentials, cosines, and sines, then we use the matrix analogue of the method of undetermined coefficients. There's also a matrix analogue of the method of variation of parameters. We'll be discussing both today.

Today's lecture corresponds with section 5.6 from the textbook, and the assigned problems are:

Section 5.6 - 1, 6, 10, 17, 19

The Method of Undetermined Coefficients

The method of undetermined coefficients for a system of ODEs works pretty much the way it did for the ODEs we studied in chapter 3. It's best understood in the context of examples.

Example - Find a particular solution to the system of ODEs:

$$\mathbf{x}' = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ t^2 \end{pmatrix}$$

Solution - We note that our function $\mathbf{f}(t)$ is a second-order polynomial with constant vector coefficients, so we guess that our particular solution will also be a second-order polynomial with constant vector coefficients:

$$\mathbf{x}_p = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2$$

If we make this guess then we have the relations:

$$\mathbf{x}'_p = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{x}_p + \begin{pmatrix} 0 \\ t^2 \end{pmatrix}.$$

Which in component form gives us:

$$\begin{pmatrix} b_1 + 2c_1t \\ b_2 + 2c_2t \end{pmatrix} = \begin{pmatrix} (3a_1 + 4a_2) + (3b_1 + 4b_2)t + (3c_1 + 4c_2)t^2 \\ (3a_1 + 2a_2) + (3b_1 + 2b_2)t + (3c_1 + 2c_2 + 1)t^2 \end{pmatrix}.$$

If we equate all the coefficients of t on both sides for each row and solve for the unknowns we get:

$$\mathbf{x}_p = \frac{1}{756} \begin{pmatrix} -868 + 840t - 504t^2 \\ 861 - 882t + 378t^2 \end{pmatrix}$$

The only issue with this method is when our function $\mathbf{f}(t)$ exhibits duplication with our homogeneous solution. So, for example, if we have the system:

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 15 \\ 4 \end{pmatrix} t e^{-2t}$$

then our homogeneous solution is:

$$\mathbf{x}_h(t) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$$

which exhibits duplication with our preliminary trial solution $\mathbf{x}_p(t) = \mathbf{a}t e^{-2t} + \mathbf{b}e^{-2t}$. Now, this is where this method differs from the earlier method of undetermined coefficients. Here, we need to multiply by t until we no longer exhibit duplication, but we still have to include all lower terms! So, our new trial solution would be:

$$\mathbf{x}_p(t) = \mathbf{a}t^2 e^{-2t} + \mathbf{b}t e^{-2t} + \mathbf{c}e^{-2t}.^1$$

We could solve this, but it turns out that the method of variation of parameters is actually simpler to use in this case. So, we'll postpone the solution and then return to it using variation of parameters.

The Method of Variation of Parameters

In matrix form the method of variation of parameters is actually pretty easy to formulate.

Suppose we have a system of ODEs:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

¹This is because we could have two linearly independent vectors $\mathbf{c}_1 e^{-2t}$ and $\mathbf{c}_2 e^{-2t}$, a situation that didn't come up in chapter 3.

and we already have a complete set of solutions to the associated homogeneous system. From these solutions we can construct a fundamental matrix $\Phi(t)$. We're going to try to write our particular solution as a linear combination of our linearly independent solutions to the homogeneous equation, but instead of being a linear combination involving constant coefficients, the coefficients will be functions. In matrix notation this is:

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{u}(t).$$

If we differentiate this we get:

$$\mathbf{x}'_p(t) = \Phi(t)'\mathbf{u}(t) + \Phi(t)\mathbf{u}'(t).$$

Now, if we use the relations:

$$\mathbf{x}'_p(t) = \mathbf{P}(t)\mathbf{x}_p + \mathbf{f}(t)$$

and

$$\Phi'(t) = \mathbf{P}(t)\Phi(t)$$

then plugging these relations into our above system and simplifying we get:

$$\mathbf{f}(t) = \Phi(t)\mathbf{u}'(t).$$

So, if we solve this for $\mathbf{u}(t)$ we get:

$$\mathbf{u}(t) = \int \Phi(t)^{-1}\mathbf{f}(t)$$

and therefore

$$\mathbf{x}_p(t) = \Phi(t) \int \Phi(t)^{-1}\mathbf{f}(t).$$

If $\mathbf{P}(t) = A$, where A is a constant matrix, then we have the fundamental matrix $\Phi(t) = e^{At}$, and our particular solution given by:

$$\mathbf{x}_p(t) = e^{At} \int e^{-At} \mathbf{f}(t) dt.$$

To solve a given initial value problem we get:

$$\mathbf{x}(t) = e^{At} \mathbf{x}_0 + e^{At} \int_0^t e^{-As} \mathbf{f}(s) ds.$$

Note the striking similarity between this equation and the corresponding equation from section 1.5.

Example - Find the general solution to the system:

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 15 \\ 4 \end{pmatrix} te^{-2t}.$$

Solution - We have from earlier the linearly independent solutions to the homogeneous equation:

$$\mathbf{x}_1 = \begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix},$$

and

$$\mathbf{x}_2 = \begin{pmatrix} 2e^{5t} \\ e^{5t} \end{pmatrix}.$$

From this we get the fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{pmatrix}.$$

The determinant of this matrix is:

$$|\Phi(t)| = \begin{vmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{vmatrix} = 7e^{3t}.$$

From this we get

$$\Phi(t)^{-1} = \frac{1}{7e^{3t}} \begin{pmatrix} e^{5t} & -2e^{5t} \\ 3e^{-2t} & e^{-2t} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} e^{2t} & -2e^{2t} \\ 3e^{-5t} & e^{-5t} \end{pmatrix}.$$

So,

$$\Phi(t)^{-1}\mathbf{f}(t) = -\frac{1}{7} \begin{pmatrix} e^{2t} & -2e^{2t} \\ 3e^{-5t} & e^{-5t} \end{pmatrix} \begin{pmatrix} 15te^{-2t} \\ 4te^{-2t} \end{pmatrix} = - \begin{pmatrix} t \\ 7te^{-7t} \end{pmatrix}.$$

Next we calculate

$$\int \Phi(t)^{-1}\mathbf{f}(t) = - \int \begin{pmatrix} t \\ 7te^{-7t} \end{pmatrix} = - \begin{pmatrix} \frac{t^2}{2} \\ -te^{-7t} - \frac{1}{7}e^{-7t} \end{pmatrix}.$$

And so, our particular solution is:

$$\mathbf{x}_p = \begin{pmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{pmatrix} \begin{pmatrix} -\frac{t^2}{2} \\ te^{-7t} + \frac{1}{7}e^{-7t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}t^2e^{-2t} + 2te^{-2t} + \frac{2}{7}e^{-2t} \\ \frac{3}{2}t^2e^{-2t} + te^{-2t} + \frac{1}{7}e^{-2t} \end{pmatrix}.$$

And, at long last, our general solution is:

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{2}t^2e^{-2t} + 2te^{-2t} + \frac{2}{7}e^{-2t} \\ \frac{3}{2}t^2e^{-2t} + te^{-2t} + \frac{1}{7}e^{-2t} \end{pmatrix} + c_1 \begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{5t} \\ e^{5t} \end{pmatrix}.$$

Easy! Just kidding.

Notes on Homework Problems

Problems 5.6.1, 5.6.6, and 5.6.10 are, increasingly more difficult, problems where you have to “guess and check” a particular solutions given the form of $\mathbf{f}(t)$. Not too difficult, although honestly a bit tedious.

Problems 5.6.17 and 5.6.19 are exercises using variation of parameters. Should be straightforward, although again kind of involved and, honestly, a bit tedious. But good for you none the less! :)