Math 2280 - Lecture 19

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Up to now all the differential equations with which we've dealt have had one dependent variable and one independent variable. So, a differential equation like:

$$y'' + 2xy' + 3e^x y = \sin x,$$

has independent variable x and dependent variable y. Today, we're going to move on to talking about *systems* of differential equations, in which there are more than one differential equation that must be satisfied, and more than one dependent variable. We will restrict our attention to systems in which the number of equations is the same as the number of dependent variables. So, for example, the system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$

where both x and y are functions of the independent variable t.

Today's lecture corresponds with section 4.1 of the textbook, and the assigned homework problems are:

Section 4.1 - 1, 2, 13, 15, 22

Systems of Equations

If we think back to linear algebra one of the major aims was solving systems of linear equations like this one:

$$\begin{array}{rcrcl} 2x & + & 3y & = & 7 \\ 3x & - & 2y & = & 4 \end{array}.$$

A solution to this system is a set of values for x and y that satisfy *both* equations. In this case the (unique) solution is $x = 2$ and $y = 1$.

For a system of differential equations we again have multiple equations that must be satisfied. For example:

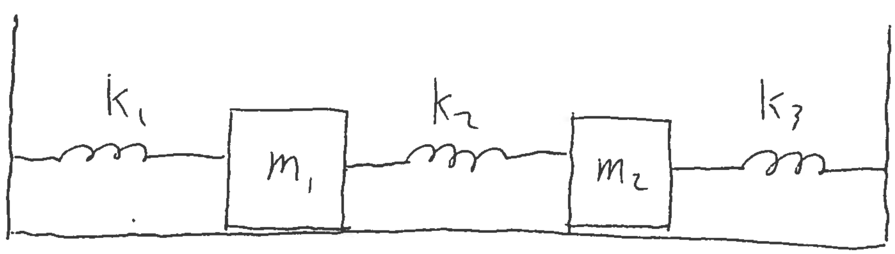
$$\begin{array}{rcrcl} x' & - & y & = & 0 \\ y' & + & x & = & 0 \end{array}.$$

A solution to this system is a pair of *functions* of the independent variable, $x(t)$ and $y(t)$, that satisfy both equations. In this case the solution is $x(t) = A\sin t + B\cos t$ and $y(t) = A\cos t - B\sin t$. Note that there are two unknown constants in this solution, A and B, and they are undetermined unless we're given values of x and y at a point a. So, we need initial conditions to determine a unique solution, but all solutions will be of the given form.

Example - Derive the equations

$$\begin{array}{rcrcl} m_1x_1'' & = & -(k_1+k_2)x_1 & + & k_2x_2 \\ m_2x_2'' & = & k_2x_1 & - & (k_2+k_3)x_2 \end{array}$$

for the displacements (from equilibrium) of the two masses shown below.



Solution - The total force on m_1 will be, from Hooke's law:

$$-k_1x_1 - k_2x_1 + k_2x_2$$
$$= -k_1x_1 - k_2(x_1 - x_2)$$
$$= -(k_1 + k_2)x_1 + k_2x_2.$$

Similarly, the force on m_2 will be:

$$k_2x_1 - (k_2 + k_3)x_2.$$

Newton's second law then gives us the relations:

$$m_1x_1'' = -(k_1 + k_2)x_1 + k_2x_2$$

and

$$m_2x_2'' = k_2x_1 - (k_2 + k_3)x_2.$$

First-Order Systems

A first-order system of differential equations is one where there are n differential equations and n dependent variables, and each equation expresses the derivative of one of the dependent variables in terms of the dependent variables and the independent variable. So, a system with independent variable t and dependent variables $x_1, \ldots, x_n$ of the form:

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \ldots, x_{n-1}, x_n) \\ x_2' &= f_2(t, x_1, x_2, \ldots, x_{n-1}, x_n) \\ &\vdots \\ x_{n-1}' &= f_{n-1}(t, x_1, x_2, \ldots, x_{n-1}, x_n) \\ x_n' &= f_n(t, x_1, x_2, \ldots, x_{n-1}, x_n) \end{aligned}$$

Every equation has the derivative of one of the dependent variables on the left, and that's the only derivative in the equation.

First, note that we can frequently transform a higher-order differential equation into a system of first-order differential equations. For example, the differential equation:

$$x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$$

can be rewritten as a system of three first-order equations by defining the variables $x_1 = x, x_2 = x_1', x_3 = x_2' = x_1''$. Then we get the system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 5x_1 - 2x_2 - 3x_3 + \sin 2t \end{aligned}$$

Almost seems like cheating, doesn't it? But, this idea is of real theoretical and practical importance. For example, remember Euler's method? Well, it only applied to first-order equations. Using this idea we can convert a higher order equation into a series of first-order equations, and apply variations of Euler's method!

Example - Transform the differential equation

$$x^{(4)} + 6x'' - 3x' + x = \cos 3t$$

into an equivalent system of first-order differential equations.

Solution - We define the new dependent variables as follows:

$$x_1 = x,$$

$$x_2 = x_1',$$

$$x_3 = x_2' = x_1'',$$

$$x_4 = x_3' = x_2'' = x_1'''$$

$$x_4' = x_3'' = x_2''' = x_1^{(4)} = -6x_1'' + 3x_1' - x_1 + \cos(3t) = -6x_3 + 3x_2 - x_1 + \cos(3t).$$

So, our system of first-order differential equations is:

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ x_3' &= x_4, \\ x_4' &= -6x_3 + 3x_2 - x_1 + \cos(3t). \end{aligned}$$

Linear Systems

In this class so far we've focused a lot of attention on linear differential equations. The main reason for this is that they're much more simple, and easy to solve, than non-linear differential equations, and so we learn them first. Also, many real world applications have linear approximations that can be useful. The same is true for systems of differential equations. A *linear* first-order system of differential equations is a system of equations with the form:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t), \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t). \end{aligned}$$

All the dependent variables on the right side appear linearly. We can rewrite this suggestively in a matrix format:

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Or, even more compactly using vectors:

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{f}.$$

Might linear algebra techniques be useful for solving systems of differential equations? We'll have to wait and see.[1]

Finally, the practical and theoretical foundations of our study of linear differential equations are the various existence and uniqueness theorems. Well, we're in luck, as there is a similar theorem for systems of linear differential equations.

Theorem - Suppose that the functions $p_{11}, p_{12}, \ldots, p_{nn}$ and the functions $f_1, f_2, \ldots, f_n$ are continuous on the open interval I containing the point a. Then given the n numbers $b_1, b_2, \ldots, b_n$, the system of differential equations above has a unique solution on the entire interval I that satisfies the n initial conditions

$$x_1(a) = b_1, x_2(a) = b_2, \ldots, x_n(a) = b_n.$$

Example - Calculate the unique solution to the given initial value problem:

$$\begin{aligned} x' &= -y \\ y' &= 13x + 4y \end{aligned}$$

$$x(0) = 0, y(0) = 3.$$

Solution - We can convert this system into a second-order linear ODE by differentiating the first equation, and plugging it into the second:

$$y = -x' \Rightarrow y' = -x'' = 13x - 4x'.$$

This gives us the second-order linear homogeneous ODE with constant coefficients:

$$x'' - 4x' + 13x = 0.$$

[1]OK, fine, the answer is yes.

The corresponding characteristic equation is $r^2 - 4r + 13$, which has roots:

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2} = 2 \pm 3i.$$

So, we get:

$$x(t) = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t),$$
$$y(t) = -x'(t) = 3Ae^{2t}\sin(3t) - 2Ae^{2t}\cos(3t) - 3Be^{2t}\cos(3t) - 2Be^{et}\sin(3t)$$
$$= (3A - 2B)e^{2t}\sin(3t) - (2A + 3B)e^{2t}\cos(3t).$$

Now, our initial conditions are $x(0) = 0$ and $y(0) = 3$. The condition $x(0) = 0$ gives us:

$$x(0) = Ae^0\cos 0 + Be^0\sin 0 = A = 0.$$

Plugging in $A = 0$ the condition $y(0) = 3$ gives us:

$$y(0) = -2Be^0\sin 0 - 3Be^0\cos 0 = -3B = 3 \Rightarrow B = -1.$$

So, our solution is:

$$x(t) = -e^{2t}\sin(3t),$$
$$y(t) = 3e^{2t}\cos(3t) + 2e^{2t}\sin(3t).$$

Notes on Homework Problems

Problems 4.1.1 and 4.1.2 are exercises in converting higher order systems to first-order systems. Should be very straightforward. Once you get the hang of how to do this (for linear systems, at any rate) it's very easy.

On problem 4.1.13 you'll want to convert this system of two first-order ODEs into an equivalent single second-order ODE that you know how to solve. It's like the last example in these notes. Problem 4.1.15 is the same.

Problem 4.1.22 uses problems 4.1.13 and 4.1.15. You'll want to show that your solutions are, respectively, circles and ellipses. Which circle and which ellipse will, of course, depend on the initial conditions.