Math 2280 - Lecture 17

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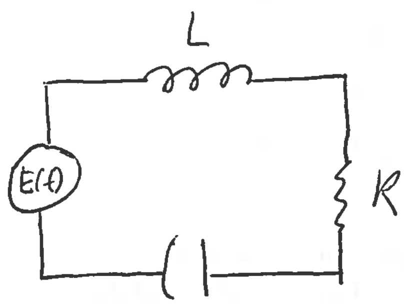
In today's lecture we'll talk about another *very* common physical system that comes up all the time in engineering - a closed circuit with a resistor, capacitor, and inductor. We'll learn that, even though physically this system is very different than a mass on a spring, the differential equation that describes them is, essentially, the same. Well, the same equations have *the same* solutions, and we'll see that the solutions we determined for the mass-spring system have their exact analogs for circuits.

Today's lecture corresponds with section 3.7 of the textbook. The assigned problems are:

Section 3.7 - 1, 5, 10, 17, 19

Electrical Circuits

For an electrical circuit of the type pictured below:



Kirchoff's second law tells us that the sum of the voltage drops across each component must equal 0:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

This is a second order linear ODE with constant coefficients! So everybody chill out, we've got this. For example, if

$$E(t) = E_0 \sin(\omega t),$$

if we differentiate both sides of the equation we get:

$$LI'' + RI' + \frac{1}{C}I = \omega E_0 \cos \omega t.$$

The homogeneous solution to this will be:

$$y_h = e^{-\frac{Rt}{2L}}\left(c_1 e^{\frac{\sqrt{R^2-4L/C}}{2L}t} + c_2 e^{-\frac{\sqrt{R^2-4L/C}}{2L}t}\right).$$

This gives us a solution for I_{tr}, the *transient* current that will die out exponentially.

The particular solution will give us another term called the *steady periodic* current. It won't die out exponentially. If we run through the math, which is exactly the same as in the mechanical system, we get:

$$I_{sp}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

where

$$\alpha = \arctan\left(\frac{\omega RC}{1 - LC\omega^2}\right).$$

The quantity in the denominator of our steady periodic current is denoted by the variable Z and is called the *impedence* of the circuit. The term $\omega L - 1/(\omega C)$ is called the *reactance*.

Using Z to denote the impedence, so

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

then the amplitude of our steady periodic current is

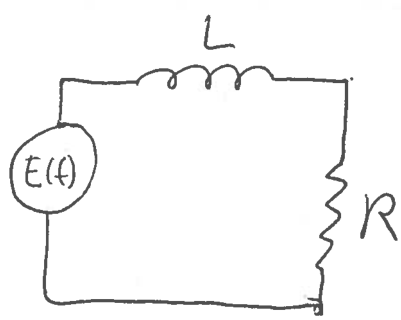
$$I_0 = \frac{E_0}{Z}.$$

If $R \neq 0$ then $Z \neq 0$, and we see that this amplitude is maximized when Z is minimized. The frequency ω that minimizes the impedence will be the frequency that makes the reactance 0. Specifically,

$$\omega_m^2 = \frac{1}{LC}.$$

This frequency is called the *resonant frequency* of the circuit.

Example - In the circuit below, suppose that $L = 2, R = 40, E(t) = 100e^{-10t}$, and $I(0) = 0$. Find the maximum current for $t \geq 0$.



Solution - If we plug the constants $L = 2, R = 40, C = 0$, and the function $E(t) = 100e^{-10t}$ into our circuit equation we get:

$$2I'(t) + 40I(t) = 100e^{-10t},$$

which simplifies to

$$I' + 20I = 50e^{-10t}.$$

This is a first-order linear differential equation, and we can solve it by multiplying both sides by the integrating factor:

$$\rho(t) = e^{\int 20dt} = e^{20t}.$$

If we do this our ODE becomes:

$$e^{20t}I'(t) + 20e^{20t}I(t) = 50e^{10t}.$$
$$\Rightarrow \frac{d}{dt}\left(Ie^{20t}\right) = 50e^{10t}.$$

Integrating both sides of the equation above gives us:

$$Ie^{20t} = \int 50e^{10t}dt = 5e^{10t} + C.$$

Solving this for the current $I(t)$ we get:

$$I(t) = 5e^{-10t} + Ce^{-20t}.$$

We have $I(0) = 0$, and we can use this to solve for C:

$$I(0) = 0 = 5 + C \Rightarrow C = -5.$$

So, the equation for our current is:

$$I(t) = 5e^{-10t} - 5e^{-20t}.$$

We want to find the value of t for which this is maximized. This will occur when the derivative is equal to 0:

$$\begin{aligned} I'(t) = -50e^{-10t} + 100e^{-20t} = 0 \\ \Rightarrow 2 = e^{10t} \\ \Rightarrow \frac{\ln 2}{10} = t. \end{aligned}$$

So, the maximum current will be:

$$I_{max}(t) = 5e^{-\ln 2} - 5e^{-2\ln 2} = \frac{5}{2} - \frac{5}{4} = \frac{5}{4} \text{ amps.}$$

Note that we can use the second derivative test to make sure this time is, indeed, a maximum:

$$I''(t) = 500e^{-10t} - 2000e^{-20t},$$

$$I''\left(\frac{\ln 2}{10}\right) = 500e^{-\ln 2} - 2000e^{-2\ln 2} = 250 - 500 = -250 < 0.$$

Example - The parameters of an RLC circuit with input voltage $E(t)$ are:

$$R = 30\Omega,\ L = 10H,\ C = 0.02F;\ E(t) = 50\sin(2t)V.$$

Substitute

$$I_{sp}(t) = A\cos\omega t + B\sin\omega t$$

using the appropriate value of ω to find the steady periodic current in the form $I_{sp}(t) = I_0\sin(\omega t - \alpha)$.

Solution - The angular frequency of our driving function is 2, so $\omega = 2$, and the amplitude I_0 will be:

$$I_0 = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{50}{\sqrt{900 + (20-25)^2}} = \frac{50}{\sqrt{925}} = \frac{10}{\sqrt{37}}.$$

As for α:

$$\alpha = arctan\left(\frac{2(30)(.02)}{1-(10)(.02)(2^2)}\right) = arctan(6).$$

So,

$$\begin{aligned} I_{sp} &= \frac{10}{\sqrt{37}}\cos(2t - arctan(6)) \\ &= \frac{10}{\sqrt{37}}\sin\left(2t - arctan(6) + \frac{\pi}{2}\right) \\ &= \frac{10}{\sqrt{37}}\sin\left(2t - (arctan(6) - \frac{\pi}{2})\right) \\ &= \frac{10}{\sqrt{37}}\sin\left(2t - (\frac{3\pi}{2} + arctan(6))\right). \end{aligned}$$

Notes on Homework Problems

Problem 3.7.1 amounts to solving a first-order linear differential equation. The hardest thing is setting it up! That's frequently the case for these problems. Problem 3.7.5 is similar, except the differential equation you need to solve is a little harder.

Problem 3.7.10 you're asked to find a steady periodic charge. This is the particular solution to the differential equtaion. The homogeneous solution dies out exponentially, so the particular solution is the one that doesn't die out, and is called "steady period" because it is, well, steady and periodic.

For problems 3.7.17 and 3.7.19 you're asked to find the behavior of an RLC circuit for different values of the constant and the driving function. The methods for solving ODEs from sections 3.3 and 3.5 will be useful to you here!