Math 2280 - Lecture 16

Dylan Zwick

Fall 2013

In today's lecture we'll return to our mass-spring mechanical system example, and examine what happens when there is a periodic driving force $f(t) = F_0 \cos \omega t$.

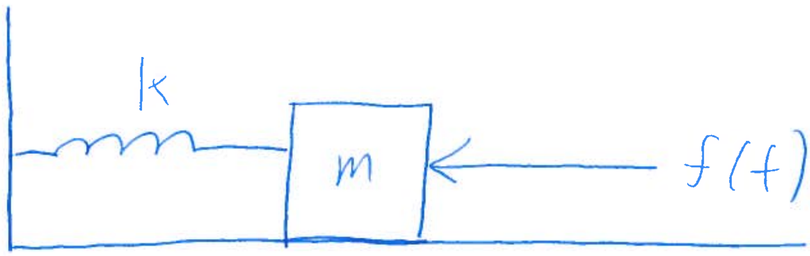
This lecture corresponds with section 3.6 of the textbook, and the assigned problems are:

Section 3.6 - 1, 2, 9, 17, 24

Forced Oscillations

In this lecture we'll delve deeper into the simple mechanical system we examined two lectures ago, and discuss some of the consequences of adding a forcing function to the system.

Suppose we have a spring-mass system with an external driving force, pictured schematically below:



Assuming there is no damping, we can model this system by a differential equation of the form:

$$mx'' + kx = f(t)$$

Now, suppose our forcing function is of the form $f(t) = F_0 \cos \omega t$, where $\omega \neq \sqrt{k/m}$. Then, the method of undetermined coefficients would lead us to guess a particular solution of the form:

$$x(t) = A \cos \omega t + B \sin \omega t.$$

If we plug this guess into our differential equation we get the relation:

$$-Am\omega^2 \cos \omega t + Ak \cos \omega t - Bm\omega^2 \sin \omega t + Bk \sin \omega t = F_0 \cos \omega t,$$

which if we solve for the constants A and B we get:

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2},$$
$$B = 0.$$

Consequently, our particular solution will be:

$$x_p(t) = \left(\frac{F_0/m}{\omega_0^2 - \omega^2} \right) \cos \omega t.$$

And, in general, our solution will be of the form:

$$x(t) = \left(\frac{F_0/m}{\omega_0^2 - \omega^2} \right) \cos \omega t + c_1 \sin \omega_0 t + c_2 \cos \omega_0 t.$$

We can, equivalently, rewrite the above solution as

$$x(t) = C \cos(\omega_0 t - \alpha) + \left(\frac{F_0/m}{\omega_0^2 - \omega^2} \right) \cos \omega t,$$

just as we did for the undamped case examined two lectures ago.

Example - Express the solution to the initial value problem

$$x'' + 9x = 10\cos 2t,$$
$$x(0) = x'(0) = 0,$$

as a sum of two oscillations as in the equation above.

Solution - The particular solution x_p will be a combination of sine and cosine terms of the form:

$$x_p = A\cos 2t + B\sin 2t.$$

Taking derivatives and plugging these into the ODE we get:

$$x_p' = -2A\sin 2t + 2B\cos 2t,$$
$$x_p'' = -4A\cos 2t - 4B\sin 2t,$$

and so,

$$x_p'' + 9x_p = 5A\cos 2t + 5B\sin 2t = 10\cos 2t.$$

So, $A = 2, B = 0$, and our particular solution is $x_p = 2\cos 2t$.

The homogeneous equation has characteristic equation $r^2 + 9$, and so the homogeneous solution is of the form:

$$x_h = c_1\sin 3t + c_2\cos 3t.$$

Our solution will be $x = x_h + x_p$, and if we plug in our initial conditions we get:

$$x(0) = c_2 + 2 = 0,$$
$$x'(0) = 3c_1 = 0,$$

from which we get $c_1 = 0$ and $c_2 = -2$. Therefore, our solution is:

$$x(t) = 2\cos 2t - 2\cos 3t.$$

Beats

If we impose the initial conditions: $x(0) = x'(0) = 0$ then we have:

$$c_1 = 0$$

and

$$c_2 = -\frac{F_0/m}{\omega_0^2 - \omega^2}.$$

Plugging these in to our solution we get:

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t).$$

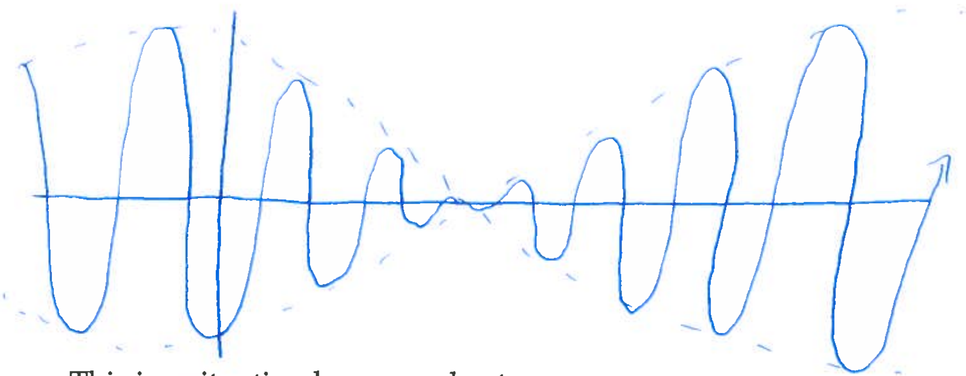
If we use the relation

$$2 \sin A \cos B = \cos (A - B) - \cos (A + B)$$

we can rewrite the above equation as:

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \left(\frac{(\omega_0 - \omega)}{2}t\right) \cos \left(\frac{(\omega_0 + \omega)}{2}t\right).$$

Now, if $\omega_0 \approx \omega$, this solution looks like a higher frequency wave oscillating within a lower frequency envelope:



This is a situation known as beats.

Resonance

What if $\omega = \omega_0$? Then, for our particular solution we'd guess:

$$x_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t).$$

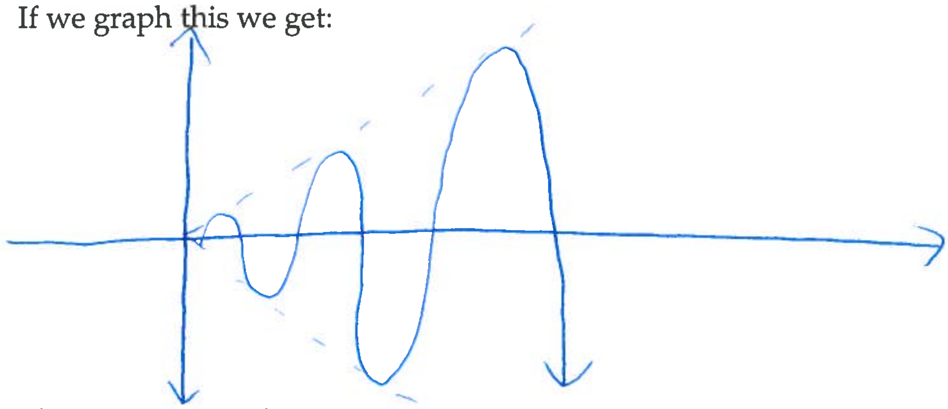
If we make this guess and work it out with the initial conditions $x(0) = x'(0) = 0$ we get:

$$A = 0$$
$$B = \frac{F_0}{2m\omega_0}$$

with corresponding particular solution:

$$x_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t).$$

If we graph this we get:



This is a situation known as resonance.

Notes on Homework Problems

On problems 3.6.1 and 3.6.2 you've asked to solve the given initial value problem and then graph the solution. The method for doing these will be the same as that used in the example problem in these lectures.

The steady periodic solution is the particular solution with no exponential decay. You'll investigate this solution in problem 3.6.9. It's what all solutions approach over time. Its frequency should be the same as the frequency of the forcing function.

For problem 3.6.17 what you should find here is that the amplitude of the steady periodic solution has a maximum at a particular value of ω, and this value of ω is what we call the "practical resonant frequency". The amplitude won't grow without bounds like in pure resonance, but it will still be pretty large.

Finally, for problem 3.6.24 the trig function $cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$ might be very useful.