Math 2280 - Lecture 1: Differential Equations - What Are They, Where Do They Come From, and What Do They Want?

Dylan Zwick

Fall 2013

Newton's fundamental discovery, the one which he considered necessary to keep secret and published only in the form of an anagram, consists of the following: *Data aequatione quotcunque fluentes quantitae involvente fluxiones invenire et vice versa*. In contemporary mathematical language, this means: "It is useful to solve differential equations".

In other words, differential equations are a big deal! One could make a reasonable argument that they are the governing equations of the universe. Unfortunately, the universe is a complicated place, and differential equations are not, in general, easy to solve.

But take heart, just because they're hard doesn't mean they're impossible, and many fairly simple and solvable differential equations do a decent job of modeling a lot of natural phenomena. Today, we'll talk a little bit about what differential equations are, and how we solve them.

Today's lecture corresponds with section 1.1 from the textbook, and the assigned problems from this section are:

Section 1.1 - 1, 12, 15, 20, 45

The Basics

A major goal in algebra, if not the major goal, is solving algebraic equations. We want to know the values of x that solve an equation like

$$x^2 - 3x + 2 = 0.$$[1]

With a differential equation, we're given a relation between a function *and its derivatives*, and we're asked to figure out what the function is. For example, we might be asked to solve

$$y' = y.$$[2]

What we want to know is the *function* $y = f(x)$ that satisfies the above relation. Now, there may be (and very frequently are) quite a few functions that will satisfy a given differential equation, but usually they will all have the same basic form. For there to be only one function that solves a differential equation in general we'll need more information, a.k.a. initial conditions.

Now, this is nothing new. You remember from calculus that if I tell you

$$\frac{dy}{dx} = 2x$$

that the solution is

$$y(x) = x^2 + C.$$

There's that $+C$ term in the answer, and any value of C will work. If we're also told the additional information $y(0) = 3$, then this specifies the value of the constant, and there is one and only one solution to our differential equation where $y(0) = 3$, namely

[1] The solutions are $x = 1$ and $x = 2$, in case you're wondering. Hopefully you know how to find these solutions!

[2] The solution is $y = Ce^x$, but we'll get to that in a little bit.

$$y(x) = x^2 + 3.$$

Example - Verify that the differential equation

$$y' + y = 0$$

is solved by any equation of the form $y(x) = Ce^{-x}$, and find the value of C such that the solution satisfies $y(0) = 2$.

We call the value of $y(0)$ the *initial condition* of our system. Technically speaking, we don't have to specify $y(0)$. We could just as well specify $y(3)$. But, for philosophical and practical reasons it's usually the value at 0 that is specified.[3]

Now, not all differential equations have solutions. The differential equation

$$(y')^2 + y^2 = -1$$

has no (real valued) solution, while the differential equation

[3]Mathematically speaking, if we know how to solve it when the value at 0 is specified, we can solve it when the value anywhere else is specified. It just involves some (frequently tedious) calculating.

$$(y')^2 + y^2 = 0$$

has only one solution, namely $y = 0$. Also, frequently we're able to prove a solution to a differential equation exists, but it can be extremely difficult to find it.

Terminology

The *order* of a differential equation is the highest derivative that appears in it. So, the differential equation

$$y' + y = 0$$

is first-order, while

$$(y'')^2 + y^2 = 0$$

is second-order. The differential equation

$$y^{(4)} + x^2 y^{(3)} + x^5 y = \sin x$$

is fourth-order. For derivatives higher than the second we denote it with an exponent inside parentheses, as in the example above. So, $y^{(8)}$ would be an eigth[4] derivative.

A *solution* to a differential equation *on the interval I* is a continuous function $u(x)$ where the derivatives $u', u'', \ldots, u^{(n)}$ exist[5] on I, and satisfy the differential equation for all x in I.

An *ordinary* differential equation is a differential equation in only one variable. We'll be almost exclusively dealing with these in this class, although we'll see some *partial* differential equations towards the end. A partial differential equation is when we have a function of more than one variable, and there is a relation between its partial derivatives.

[4] yikes!
[5] where n is the order of the differential equation.

If we're given initial values for a differential equation, a solution to the *initial value problem* will be a function that satisfies the differential equation in an interval around the initial value, and is equal to the initial value at the appropriate point.

Example - Given the solution $y(x) = \frac{1}{C - x}$ to the differential equation $y' = y^2$ solve the initial value problem

$$\frac{dy}{dx} = y^2, \qquad\qquad y(1) = 2.$$

Notes on Homework Problems

The first two homework problems (1.1.1, 1.1.12) just ask you to verify that certain functions are, indeed, solutions to a given differential equation. Just like the examples we've done above. Shouldn't be too hard.

The third problem (1.1.15) is similar, but it hints at a method for solving a type of differential equation called a *linear* differential equation that you'll be seeing many, many times in this course. Pay attention to this one, because you will see it again.

The fourth problem (1.1.20) is a simple initial value problem. Again, shouldn't be too bad.

The fifth and final problem (1.1.45) is a lot of fun! It's an example of something called a *doomsday* equation, and when you work the problem you'll find out why. Note that in order to solve this problem you'll need a result from an earlier problem (1.1.43) that says that the solution to a differential equation of this type has the form:

$$P(t) = \frac{1}{(C - kt)}.$$

You're not expected to know how to find that solution *yet*, although in a couple of weeks you'll be able to do so!